2026–2027 OFFICIAL COURSEBOOK EDITION

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AP Physics C: Mechanics Complete Master Textbook

Units 1–7 Calculus-Based Theory, Derivations, Reference Tables & Step-by-Step Worked Problems

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Energy & Momentum of Rotating Systems, Gravitation & Orbits

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Oscillations & Simple Harmonic Motion

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CHAPTER 1 · CORE CURRICULUM

AP Exam Weighting: 14–20%

Unit 1: Kinematics & Calculus Foundations

Classical mechanics rests upon three fundamental SI base quantities—length (meters), time (seconds), and mass (kilograms)—coupled with vector algebra and differential/integral calculus. This unit establishes experimental measurement standards, vector decomposition, the calculus ladder linking position x(t), velocity v(t), and acceleration a(t), uniformly accelerated motion (UAM), two-dimensional projectile motion, and relative velocity across inertial reference frames.

CHAPTER 1 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$E_r = ((O - A) / A) \times 100\%$	Relative error (quantitative measure of experimental accuracy)
$v(t) = dx/dt \mid a(t) = dv/dt = d^2x/dt^2$	Instantaneous velocity and acceleration via differentiation
$\Delta v = \int a(t)\, dt \mid \Delta x = \int v(t)\, dt$	Definite integral changes from signed area under motion curves
$f_{avg} = [1 / (t_2 - t_1)] \int (t_1 \text{ to } t_2)\, f(t)\, dt$	Calculus average value of a continuous time-varying function
$v_x = v_{x0} + a_x t \mid x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2 \mid v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$	Uniformly Accelerated Motion (UAM) kinematic equations (constant a_x only)
$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$ (with $\vec{v}_{AB} = -\vec{v}_{BA}$)	Relative velocity transformation across inertial reference frames

§ 1.1 SI Base Units, Significant Figures, Accuracy & Precision

Every physical measurement in classical mechanics is constructed from just three fundamental quantities and their **SI base units**: length or distance measured in **meters (m)**, time measured in **seconds (s)**, and mass measured in **kilograms (kg)**. Every derived unit in mechanics is an algebraic combination of meters, seconds, and kilograms. When converting units, multiply by conversion factors equal to unity (1) so undesired units cancel cleanly across the numerator and denominator.

Significant figures represent all digits actually read from a measuring instrument's scale plus one final estimated digit. For example, if the tip of a marker lies just before 132 mm on a millimeter ruler, we record **131.9 mm** (four significant figures), where the final 0.9 mm is the estimated tenth of a millimeter.

In experimental physics, **accuracy** and **precision** describe two distinct properties. **Accuracy** measures how close an observed value O is to the accepted reference value A (such as Earth's standard free-fall acceleration g = 9.81 m/s^2), and is quantified by **relative error**: $E_r = ((O - A) / A) \times 100\%$. Because O and A carry identical units, the units cancel completely, leaving relative error as a dimensionless percentage. **Precision** measures how closely repeated measurements cluster around one another (experimental repeatability), quantified by the spread or standard deviation of repeated trials around their mean.

- **Rule 1 (Non-zero Digits):** All non-zero digits are always significant.
- **Rule 2 (Captive Zeros):** Zeros between significant digits are always significant (e.g., 70.2 has 3 sig figs).
- **Rule 3 (Leading Zeros):** Leading zeros to the left of the first non-zero digit are never significant (e.g., 0.045 has 2 sig figs).
- **Rule 4 (Trailing Zeros):** Trailing zeros at the end of a number are significant only if they lie to the right of a decimal point (e.g., 70.0 has 3 sig figs, whereas 100 has 1 sig fig; write 1.20×10^3 for 1,200 with 3 sig figs).
- **Target Analogy (4 Cases):** (1) Tight cluster in bullseye = high accuracy, high precision; (2) Tight cluster far from bullseye = low accuracy, high precision (classic systematic error); (3) Wide spread whose geometric mean hits bullseye = high average accuracy, low precision; (4) Single arrow in bullseye = high accuracy, precision undetermined (requires multiple trials).

Table 1.1 · Fundamental and Derived SI Units in Classical Mechanics

Physical Quantity	Symbol	SI Unit Name	Base Unit Equivalent
Length / Position	x, r, L	meter (m)	m
Time	t	second (s)	s
Mass	m, M	kilogram (kg)	kg
Velocity / Speed	v	meter per second	$m \cdot s^{-1}$ (m/s)
Acceleration	a	meter per second squared	$m \cdot s^{-2}$ (m/s^2)
Force / Weight	F, F_g	newton (N)	$kg \cdot m \cdot s^{-2}$ ($kg \cdot m/s^2$)
Energy / Work	E, K, U, W	joule (J)	$kg \cdot m^2 \cdot s^{-2}$ (N·m)

EXAMPLE 1.1 Quantifying Experimental Accuracy via Relative Error

> **Problem:** In a free-fall drop experiment, a student measures local gravitational acceleration as O = 9.61 m/s^2, compared to the accepted value A = 9.81 m/s^2. Calculate the relative error and interpret its sign.

STEP-BY-STEP SOLUTION:

1. Step 1: Write the relative error formula: E_r = ((O - A) / A) × 100%.
2. Step 2: Substitute O = 9.61 m/s^2 and A = 9.81 m/s^2: E_r = ((9.61 m/s^2 - 9.81 m/s^2) / 9.81 m/s^2) × 100%.
3. Step 3: Notice that the units m/s^2 cancel between numerator and denominator: E_r = (-0.20 / 9.81) × 100% = -2.04%.

Final Answer: E_r = -2.04% (the measured value is 2.04% below the accepted value; this measures accuracy, not precision).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Relative error strictly measures accuracy (comparison against the accepted value A), never precision. Furthermore, a single measurement can be accurate, but its precision is undefined without repeated trials.

§ 1.2 Scalars, Vectors, Unit Vectors & Calculus Differentials

Physical quantities divide into **scalars** and **vectors**. A **scalar** possesses magnitude only with no direction (such as mass, time, distance, speed, and energy). A **vector** possesses both magnitude and direction (such as displacement, velocity, acceleration, force, and momentum). To represent direction algebraically, we use dimensionless **unit vectors**: **î (or x̂)** points one unit along the +x-axis, and **ĵ (or ŷ)** points one unit along the +y-axis.

When a two-dimensional velocity vector **v** of magnitude v is launched at angle θ above the horizontal, right-triangle trigonometry resolves it into orthogonal components: $\mathbf{v_x = v \cos \theta}$ along î and $\mathbf{v_y = v \sin \theta}$ along ĵ, with total magnitude given by the Pythagorean theorem: $\mathbf{v = \sqrt{(v_x^2 + v_y^2)}}$.

In AP Physics C, physical units immediately reveal the calculus structure of any quantity. Whenever a unit contains a division bar, it represents a **rate of change (a derivative)**: m/s identifies velocity v = dx/dt; honey poured at 0.8 kg/hr identifies the time derivative of mass dm/dt; a non-uniform rod whose linear density increases at 4 kg/m identifies the spatial derivative of mass **λ = dm/dx**. Conversely, a **differential** (dx, dt, dm, dA) represents an infinitesimally thin slice of a quantity, and **integration (∫)** sums infinitely many slices either over time (signed area under a rate curve, Δx = ∫ v(t) dt) or over space (total mass M = ∫ dm, moment of inertia I = ∫ r^2 dm).

- **Ratio of Differentials:** Viewing dx/dt = 4 as a ratio of differentials means the position differential dx grows 4 times as fast as the time differential dt.

- **Concentric Ring Integration:** Slicing a circle of radius R into thin concentric rings of radius r and thickness dr gives unrolled strip length $2\pi r$ and differential area $dA = 2\pi r\, dr$; integrating $\int_0^R 2\pi r\, dr = [2\pi r^2/2]_0^R = \pi R^2$.

EXAMPLE 1.2 **Distance vs. Displacement and Average Speed vs. Average Velocity**

> **Problem:** A runner sprints 50 m east and then walks 30 m west in a total elapsed time of 10 s. Determine (a) the total distance and net displacement, and (b) the average speed and average velocity.

STEP-BY-STEP SOLUTION:

1. Step 1: Distance is the scalar total path length: d = 50 m + 30 m = 80 m. Displacement is the net vector change $\Delta x = x_f - x_0$ = +50 m - 30 m = +20 m east.
2. Step 2: Average speed = total distance / total time = 80 m / 10 s = 8.0 m/s.
3. Step 3: Average velocity = net displacement / elapsed time = +20 m east / 10 s = +2.0 m/s east.

Final Answer: Distance = 80 m, Displacement = +20 m east; Average speed = 8.0 m/s, Average velocity = +2.0 m/s east.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Average speed is NOT the magnitude of average velocity whenever an object reverses direction! To find total distance traveled from a v(t) graph, sum the absolute values of all areas above and below the time axis: Distance = $\int |v(t)|\, dt$.

§ 1.3 The Calculus Kinematics Ladder & Time-Varying Acceleration

On a position-time graph x(t), the slope of a **secant line** connecting (t_1, x_1) and (t_2, x_2) gives average velocity $v_{avg} = \Delta x/\Delta t$. Shrinking $\Delta t \to 0$ turns the secant line into the **tangent line** at a single instant, defining instantaneous velocity $\mathbf{v(t) = dx/dt}$. Differentiating a second time defines instantaneous acceleration: $\mathbf{a(t) = dv/dt = d^2x/dt^2}$.

Position x(t), velocity v(t), and acceleration a(t) form a complete **calculus ladder**. Moving *down* the ladder (x → v → a), we differentiate by reading the instantaneous slope of the curve. Moving *up* the ladder (a → v → x), we integrate by computing the **signed area** bounded between the curve and the time axis: the signed area under a(t) from t_1 to t_2 equals $\mathbf{\Delta v = v(t_2) - v(t_1)}$, and the signed area under v(t) from t_1 to t_2 equals net displacement $\mathbf{\Delta x = x(t_2) - x(t_1)}$.

When acceleration is a non-constant function of time a(t), we find average acceleration over $[t_1, t_2]$ using the **integral average value theorem**: $a_{avg} = [1/(t_2 - t_1)] \int_{t_1}^{t_2} a(t)\, dt = \Delta v/\Delta t$.

- **Golden Sign Rule of Speeding Up / Slowing Down:** An object speeds up whenever velocity v and acceleration a have the SAME sign ($v \cdot a > 0$), and slows down whenever v and a have OPPOSITE signs ($v \cdot a < 0$).
- **Reading x(t) Curvature:** Where x(t) slopes upward, v > 0; at a horizontal peak or trough, v = 0; where x(t) slopes downward, v < 0; concave up means a > 0, and concave down means a < 0.

Table 1.2 · The Kinematic Calculus Ladder & Graphical Interpretations

Graph Type	Instantaneous Value (y-coord)	Slope (Derivative d/dt)	Signed Area Under Curve (Integral ∫ dt)
Position vs. Time x(t)	Position x	Instantaneous Velocity v(t) = dx/dt	Not a standard mechanical quantity
Velocity vs. Time v(t)	Velocity v	Instantaneous Acceleration a(t) = dv/dt	Net Displacement $\Delta x = \int v(t)\, dt$
Acceleration vs. Time a(t)	Acceleration a	Jerk da/dt	Change in Velocity $\Delta v = \int a(t)\, dt$

EXAMPLE 1.3 **Average Acceleration of a Time-Varying Acceleration Function**

Problem: A particle experiences a time-varying acceleration given by $a(t) = 4t$ m/s^2. Determine the particle's average acceleration over the time interval from $t = 0$ to $t = 4$ s.

STEP-BY-STEP SOLUTION:

1. Step 1: Apply the calculus average value formula over [a, b] = [0, 4]: a_avg = [1 / (4 - 0)] ∫(0 to 4) 4t dt.
2. Step 2: Evaluate the antiderivative of 4t, which is $2t^2$: ∫(0 to 4) 4t dt = [$2t^2$](0 to 4) = $2(4)^2$ - 0 = 32 m/s (this equals Δv).
3. Step 3: Divide Δv = 32 m/s by the time interval Δt = 4 s: a_avg = 32 / 4 = 8 m/s^2.

Final Answer: a_avg = 8 m/s^2.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never use the constant-acceleration UAM equations ($v = v_0 + at$ or $\Delta x = v_0t + \frac{1}{2}at^2$) when acceleration varies with time a(t) or velocity a(v)—you MUST integrate!

§ 1.4 Uniformly Accelerated Motion (UAM), Free Fall & 2D Projectiles

When acceleration a_x is strictly constant, integrating $dv/dt = a_x$ yields $\mathbf{v_x = v_{x0} + a_x t}$ (missing Δx). Integrating a second time yields $\mathbf{x = x_0 + v_{x0} t + \frac{1}{2}a_x t^2}$ (missing final v_x). Eliminating time t between the two equations produces the time-independent relation $\mathbf{v_x^2 = v_{x0}^2 + 2a_x(x - x_0)}$ (missing t). Memorizing which variable is absent from each UAM equation allows rapid selection on timed exams.

An object is in **free fall** whenever gravity is the sole force acting upon it (air resistance negligible). Near Earth's surface, all free-falling objects—whether a heavy hammer or a feather in a vacuum—experience the exact same downward acceleration: $\mathbf{a_y = -g = -9.81\ m/s^2\ (\approx -10\ m/s^2)}$. When a ball is thrown straight upward with initial velocity $+v_0$, it slows at 9.81 m/s^2 on the way up, reaches instantaneous vertical velocity $\mathbf{v_y = 0}$ **at the apex** while its acceleration remains $\mathbf{-9.81\ m/s^2}$ **downward**, and returns to launch height in an equal fall time with impact speed equal to launch speed.

In **two-dimensional projectile motion**, horizontal and vertical motions are completely independent and share only the common variable **time t**. Launching at speed v_0 and angle θ above the horizontal yields initial components $\mathbf{v_{x0} = v_0 \cos\theta}$ and $\mathbf{v_{y0} = v_0 \sin\theta}$. With no air resistance, horizontal acceleration is $a_x = 0$, so v_x remains constant and $\mathbf{\Delta x = (v_0 \cos\theta)t}$. Vertically, constant downward acceleration $a_y = -g$ causes v_y to decrease linearly and zero out at the apex while vertical displacement follows the parabola $\mathbf{\Delta y = (v_0 \sin\theta)t - \frac{1}{2}gt^2}$.

- **Apex Condition:** At the highest point of a 2D projectile trajectory, $v_y = 0$, total speed is minimum ($v_{min} = v_{x0} = v_0 \cos\theta$), and acceleration is still $\vec{a} = -g\hat{j}$ downward.
- **Trajectory Symmetry:** For level-ground launches (Δy = 0), ascent time equals descent time ($t_{up} = t_{down} = v_0 \sin\theta / g$), and launch speed equals impact speed.
- **Relative Velocity ($\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$):** By the inner-subscript cancellation rule, when B is the inner subscript on the right-hand side, the vectors add to give A relative to C, and reversing any subscript flips the sign ($\vec{v}_{AB} = -\vec{v}_{BA}$).
- **2D River Crossing:** When a boat with speed v_{BW} relative to the water heads straight across a river of width W while the current flows downstream at v_{WG}, crossing time depends solely on the perpendicular speed across the river ($t = W / v_{BW}$), while downstream drift is $\Delta x = v_{WG} \cdot t$.

EXAMPLE 1.4 2D Projectile Motion with Variable Headwind Acceleration

Problem: A cannon fires a ball from level ground at angle $\theta = 30°$ with initial speed $v_0 = 10$ m/s (use $g = 10$ m/s^2). Simultaneously, an opposing headwind produces a variable horizontal acceleration $a_x(t) = -t^2/4$ m/s^2. How far horizontally does the ball travel before striking the ground?

STEP-BY-STEP SOLUTION:

1. Step 1 (Vertical Flight Time): Initial vertical velocity is $v_{y0} = 10 \sin 30° = 5.0$ m/s. Setting $\Delta y = v_{y0} t - \frac{1}{2}gt^2 = 0$ gives $0 = 5t - 5t^2$, so the total flight time is $t = 1.0$ s.
2. Step 2 (Initial Horizontal Velocity): $v_{x0} = 10 \cos 30° \approx 8.66$ m/s.
3. Step 3 (Variable Horizontal Acceleration Integration): Integrate $a_x(t) = -t^2/4$ twice (or integrate $v_x(t) = v_{x0} - t^3/12$ from $t = 0$ to $t = 1.0$ s): $\Delta x = \int_0^1 (8.66 - t^3/12)\, dt = [8.66t - t^4/48]_0^1 = 8.66 - 1/48 \approx 8.64$ m (or using average acceleration $a_{avg} = -1/12$ m/s^2 as approximated in lecture: $\Delta x \approx 8.66(1) + \frac{1}{2}(-1/12)(1)^2 = 8.66 - 1/24 = 8.62$ m).

Final Answer: Flight time $t = 1.0$ s; horizontal range $\Delta x \approx 8.62$–8.64 m.

⚠ AP EXAM TRAP & FRQ SCORING GUIDELINE

AP Exam Trap: Never state that acceleration is zero at the peak (apex) of free fall or projectile motion! At the apex, only vertical velocity v_y is zero; gravitational acceleration is still 9.81 m/s^2 downward.

✅ CHAPTER 1 MASTERY CHECKLIST (AP EXAM READINESS)

☑ Convert derived units to SI base units (kg, m, s) and distinguish experimental accuracy (relative error) from precision (repeatability).

☑ Climb up and down the kinematic calculus ladder: differentiate $x(t) \to v(t) \to a(t)$ via slope and integrate $a(t) \to v(t) \to x(t)$ via signed area.

☑ Apply the integral average value formula $f_{avg} = [1/(b-a)] \int f(t)\, dt$ whenever acceleration varies with time.

☑ Decouple 2D projectile motion into horizontal ($a_x = 0$) and vertical ($a_y = -g$) components linked by time t, and apply inner-subscript cancellation for relative velocity.

CHAPTER 2 · CORE CURRICULUM

AP Exam Weighting: 17–23%

Unit 2: Newton's Laws of Motion & Dynamics

Dynamics investigates the causal relationship between forces and motion. This unit covers discrete and continuous center-of-mass determinations via calculus, free-body diagram (FBD) construction, Newton's three laws of motion, universal gravitation and Newton's Shell Theorem for radially varying planets, static and kinetic friction on inclined planes, coupled Atwood systems, first-order differential equations for velocity-dependent resistive drag, Hooke's law for spring combinations, and uniform circular motion.

CHAPTER 2 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$x_{cm} = (\sum m_i x_i) / M = (1/M) \int x\, dm$	Center of mass for discrete point particles and continuous bodies ($dm = \lambda(x)\, dx$)
$\sum \vec{F} = m \vec{a} = m (d\vec{v}/dt) \mid \vec{F}_{A \text{ on } B} = -\vec{F}_{B \text{ on } A}$	Newton's Second Law of Motion and Third Law action-reaction pair symmetry
$F_g = G m_1 m_2 / r^2 \mid M_{enc}(r) = \int_{0 \text{ to } r} \rho(r') 4\pi r'^2 dr'$	Universal gravitation and enclosed mass via Newton's Shell Theorem
$f_s \le \mu_s F_N \mid f_k = \mu_k F_N \mid \vec{F}_s = -k \Delta \vec{x}$	Static/kinetic friction laws and Hooke's linear restoring spring law
$m (dv/dt) = mg - kv \Rightarrow v(t) = (mg/k)(1 - e^{(-kt/m)})$	Resistive fluid drag differential equation and exponential terminal velocity solution
$a_c = v^2 / r = \omega^2 r \mid \tan\theta = v^2 / (rg)$	Centripetal acceleration and ideal frictionless banked curve relation

§ 2.1 Center of Mass (Discrete & Continuous) and Free-Body Diagrams

The **center of mass (CM)** of a system is the single mass-weighted average point where the entire mass of the system can be treated as concentrated when external forces act upon it. For a discrete system of point masses, **$x_{cm} = (\sum m_i x_i) / M_{total}$**. For example, if two 10 kg spheres sit at x = 0 m and one 10 kg sphere sits at x = 1 m, then $x_{cm} = (0 + 0 + 10 \cdot 1) / 30 = 1/3$ m.

For a continuous object such as a non-uniform rod, the summation becomes a definite ratio of integrals: **$x_{cm} = (\int x\, dm) / (\int dm)$**. Because we cannot integrate x directly with respect to dm, we substitute **$dm = \lambda(x)\, dx$**, where **$\lambda(x) = dm/dx$** is the linear mass density in kg/m.

A **free-body diagram (FBD)** models a translational object as a single center-of-mass dot and displays every external force acting directly on that object. On the AP Physics C exam, four strict rules govern every FBD: (1) every force arrow must start on the dot and point outward in the direction the force acts; (2) draw *only* actual physical forces (gravity F_g, normal F_N, tension T, applied force F_{app}, friction f)—**never draw velocity v, acceleration a, or net force ∑F on the FBD itself**; (3) resolve forces into components only on a separate working diagram if the exam grades raw force vectors; and (4) draw arrow lengths proportional to magnitude.

- **Linear, Surface & Volumetric Densities:** For 1D rods use $dm = \lambda(x)\, dx$; for 2D thin plates/rings use $dm = \sigma\, dA$; for 3D solid spheres use $dm = \rho(r)\, dV = \rho(r)\, 4\pi r^2\, dr$.
- **Proportional FBD Arrows:** For a stationary block on a horizontal floor, the upward F_N arrow and downward $F_g = mg$ arrow must have identical lengths.

EXAMPLE 2.1 **Center of Mass of a Non-Uniform Rod via Integration**

Problem: A non-uniform rod of length L extends along the x-axis from x = 0 to x = L and has a position-dependent linear mass density $\lambda(x) = 4x^2$. Find the x-coordinate of its center of mass in terms of L.

STEP-BY-STEP SOLUTION:

1. Step 1: Express the differential mass element dm in terms of dx: $dm = \lambda(x)\,dx = 4x^2\,dx$.
2. Step 2: Integrate to find the total mass M in the denominator: $M = \int_0^L dm = \int_0^L 4x^2\,dx = [4x^3/3]_0^L = (4/3)L^3$.
3. Step 3: Integrate the first mass moment in the numerator: $\int_0^L x\,dm = \int_0^L x(4x^2\,dx) = \int_0^L 4x^3\,dx = [x^4]_0^L = L^4$.
4. Step 4: Divide numerator by denominator: $x_{cm} = L^4 / ((4/3)L^3) = (3/4)L$.

Final Answer: x_{cm} = ¾L (shifted toward the denser right end of the rod).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: When finding the center of mass of a non-uniform rod whose density λ(x) is given, do NOT divide by an ungiven symbol M if the answer must be in terms of L—compute $M = \int_0^L \lambda(x)\,dx$ explicitly!

§ 2.2 Newton's Three Laws, Universal Gravitation & Shell Theorem

Newton's First Law states that an object at rest remains at rest, and an object in motion continues in straight-line motion at constant velocity, unless acted upon by a net external force. **Inertia** is an object's natural resistance to changes in velocity, and **mass (m)** is the quantitative scalar measure of inertia. Translational equilibrium occurs whenever **$\sum \vec{F} = 0$** (both $\sum F_x = 0$ and $\sum F_y = 0$), which means the force vectors form a closed tip-to-tail polygon; this includes both static rest (v = 0) and dynamic equilibrium (sliding at constant velocity).

Newton's Second Law links net external force to acceleration: **$\sum \vec{F} = m\vec{a} = m(d\vec{v}/dt)$**, applied independently along orthogonal axes ($\sum F_x = ma_x$, $\sum F_y = ma_y$). In an accelerating elevator, a bathroom scale measures the upward normal force **$F_N = m(g + a_y)$** (taking upward as +y): you feel heavier when accelerating upward ($a_y > 0$), lighter when accelerating downward ($a_y < 0$), and apparently weightless in free fall ($a_y = -g \Rightarrow F_N = 0$). **Newton's Third Law** states that forces always arise in simultaneous interaction pairs between two objects: **$\vec{F}_{A \text{ on } B} = -\vec{F}_{B \text{ on } A}$**. Whether a massive truck hits a small car or Earth pulls on the Moon, the interaction forces are strictly equal in magnitude; the smaller mass simply experiences a much larger acceleration (a = F/m). Action-reaction pairs never cancel because they act on two different objects.

Newton's Law of Universal Gravitation states that every mass attracts every other mass with magnitude **$F_g = G m_1 m_2 / r^2$**, where r is the center-to-center distance. Equating $mg = GMm/R^2$ at a planet's surface yields the local gravitational field **$g = GM/R^2$**. If a particle is located *inside* a planet at radius r < R, **Newton's Shell Theorem** proves that all spherical mass shells outside radius r cancel to zero net force; only the enclosed mass **$M_{enc}(r) = \int_0^r \rho(r')\,4\pi r'^2\,dr'$** pulls inward, giving $F_g(r) = G\,M_{enc}(r)\,m / r^2$.

- **Mass vs. Weight:** Mass (kg) is an intrinsic scalar measuring inertia that never changes with location; weight $F_g = mg$ (N) is the gravitational force vector that varies with local g.
- **Book on a Table:** For a book resting on a table, F_N and F_g are equal and opposite by Newton's First Law (both act on the book), NOT a Newton's Third Law pair! The Third Law partner to Earth pulling down on the book is the book pulling gravitationally up on Earth.

EXAMPLE 2.2 **Gravity Inside a Non-Uniform Planet via Newton's Shell Theorem**

Problem: A spherical planet of radius R has a radially varying volumetric mass density $\rho(r) = br$, where b is a positive constant. Derive an expression for the gravitational force F_g acting on a test mass m located inside a vertical tunnel at distance $r < R$ from the planet's center.

STEP-BY-STEP SOLUTION:

1. Step 1: Slice the interior sphere into concentric thin spherical shells of radius r', surface area $4\pi r'^2$, thickness dr', and volume $dV = 4\pi r'^2 dr'$.
2. Step 2: Write the differential mass of one shell: $dm = \rho(r')\, dV = (br')(4\pi r'^2 dr') = 4\pi b\, r'^3 dr'$.
3. Step 3: Integrate from 0 to r to find the enclosed mass: $M_{enc}(r) = \int_0^r 4\pi b\, r'^3 dr' = 4\pi b\, [r'^4/4]_0^r = \pi b\, r^4$.
4. Step 4: By Newton's Shell Theorem, outer shells ($r' > r$) exert zero net force. Substitute $M_{enc}(r)$ into universal gravitation: $F_g(r) = G\, M_{enc}(r)\, m / r^2 = G(\pi b\, r^4)m / r^2 = \pi G b m\, r^2$.

Final Answer: $F_g(r) = \pi G b m\, r^2$ directed radially inward.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never use surface-to-surface altitude h alone in $F_g = GMm/r^2$; the denominator r is always measured from center of mass to center of mass ($r = R_{planet} + h$).

§ 2.3 Friction, Inclined Planes & Connected Atwood Systems

Friction acts parallel to a contact surface to oppose relative sliding motion or impending sliding. **Static friction** satisfies the inequality $\mathbf{f_s \le \mu_s F_N}$: it is a self-adjusting force that matches an applied push one-for-one up to the breakaway threshold $\mathbf{f_{s,max} = \mu_s F_N}$. Once sliding begins, friction drops to a constant **kinetic friction** value $\mathbf{f_k = \mu_k F_N}$ (where $\mu_k < \mu_s$). When holding a block against a vertical wall by pushing horizontally with F_{push}, the normal force is $F_N = F_{push}$ while upward static friction balances weight ($f_s = mg$).

For a block of mass m on an **inclined plane** tilted at angle θ above the horizontal, always tilt the coordinate axes so the x-axis runs parallel to the ramp and the y-axis runs perpendicular to the ramp. Because F_N and friction f already lie along the tilted axes, only gravity mg must be resolved. By similar right triangles, the angle between straight-down mg and the perpendicular y-axis is θ, yielding perpendicular component **mg cos θ** (so $F_N = mg\cos\theta$) and parallel component **mg sin θ** pulling down the ramp. Sliding down a rough incline, $\sum F_x = mg\sin\theta - \mu_k mg\cos\theta = ma$ gives mass-independent acceleration $\mathbf{a = g(\sin\theta - \mu_k \cos\theta)}$.

For **connected multi-body systems** (such as an **Atwood machine** with masses m_1 and m_2 connected by a massless inextensible cord over a frictionless, massless pulley), use the two-step *System + Individual FBD* method: (1) treat both blocks as a single system of mass $(m_1 + m_2)$ moving along the rope line so internal tensions +T and -T cancel, giving $\mathbf{a = (m_2 - m_1)g / (m_1 + m_2)}$; (2) isolate a single block on its own FBD and substitute a to solve for internal tension T.

- **Critical Angle of Repose:** A block on a rough incline just begins to slip when $mg\sin\theta_c = \mu_s mg\cos\theta_c$, or $\tan\theta_c = \mu_s$.
- **Atwood Tension Check:** In an accelerating Atwood machine ($m_2 > m_1$), the rope tension $T = 2m_1m_2g / (m_1 + m_2)$ always lies strictly between the two weights: $m_1g < T < m_2g$.

Table 2.1 · Normal Force F_N Across Common Mechanical Configurations

Physical Scenario	Vertical / Perpendicular Balance	Normal Force Expression F_N
Block on flat horizontal surface (no vertical push)	F_N - mg = 0	F_N = mg
Elevator accelerating vertically at a_y (upward +)	F_N - mg = m a_y	F_N = m(g + a_y)
Block pinned against vertical wall by horizontal F_push	F_N - F_push = 0	F_N = F_push
Block on inclined plane at angle θ	F_N - mg cos θ = 0	F_N = mg cos θ

EXAMPLE 2.3 **System + Individual FBD Analysis of an Atwood Machine**

> **Problem:** Two blocks of masses m_1 = 2.0 kg and m_2 = 6.0 kg hang from ends of a massless cord over a frictionless, massless pulley (use g = 10 m/s^2). Find (a) the magnitude of the blocks' acceleration and (b) the tension T in the cord.

STEP-BY-STEP SOLUTION:

1. Step 1 (System Equation): Choose the path of motion (upward on m_1, downward on m_2) as positive. Internal tensions cancel: $\sum F_{sys} = m_2 g - m_1 g = (m_1 + m_2)a$.
2. Step 2 (Solve for a): a = (6.0 - 2.0)(10) / (2.0 + 6.0) = 40 / 8.0 = 5.0 m/s^2.
3. Step 3 (Isolate m_1 for Tension): From m_1's FBD, $T - m_1 g = m_1 a \Rightarrow T = m_1(g + a)$ = 2.0(10 + 5.0) = 30 N.

Final Answer: a = 5.0 m/s^2; T = 30 N.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never blindly substitute F_N = mg into the friction equation f = μF_N! On an incline F_N = mg cos θ, and against a vertical wall F_N = F_push.

§ 2.4 Resistive Drag Differential Equations, Springs & Circular Motion

One of the highest-yield calculus topics on the AP Physics C exam is motion with a **velocity-dependent resistive drag force**, **F⃗_drag = -kv⃗** (or -bv⃗), where k > 0 and the minus sign indicates that drag opposes instantaneous velocity. When an object of mass m is dropped from rest (taking downward as positive), initial velocity is v(0) = 0 so initial acceleration is a(0) = g. As speed increases, upward drag kv grows, reducing net downward force: **m(dv/dt) = mg - kv**. Eventually kv balances mg completely, acceleration drops to zero, and the object falls at constant **terminal velocity v_T = mg/k**.

An ideal spring exerts a linear restoring force governed by **Hooke's Law: F⃗_s = -k Δx⃗**, where k (in N/m) is the spring constant (the slope of an |F_s| vs. |Δx| graph). When two springs k_1 and k_2 are connected side-by-side in **parallel**, they share the same stretch Δx and their stiffnesses add: **$k_p = k_1 + k_2$**. When connected end-to-end in **series**, they share the same tension and their extensions add, yielding the reciprocal rule: **$1/k_s = 1/k_1 + 1/k_2$**.

When an object moves in a circle of radius r at constant speed v, it is continuously accelerating because the *direction* of its tangent velocity vector v⃗ is constantly turning. The **centripetal acceleration** points radially inward toward the center with magnitude **$a_c = v^2/r = \omega^2 r$**, where angular velocity is ω = dθ/dt = 2π/T = 2πf and tangential speed is v = rω. By Newton's Second Law, net inward radial force must equal **$\sum F_r = mv^2/r$**. On a flat curve of radius R, static friction provides the inward force, limiting safe speed to **$v_{max} = \sqrt{\mu_s g R}$**. On a frictionless curve banked at angle θ, horizontal normal component F_N sin θ provides mv^2/R while vertical component F_N cos θ balances mg; dividing the two equations cancels m and F_N to yield **$\tan\theta = v^2/(Rg)$**.

- **Horizontal Drag Decay:** For an object coasting horizontally under drag alone, m(dv/dt) = -kv ⇒ ∫(v_0 to v) dv/v = ∫(0 to t) (-k/m) dt ⇒ ln(v/v_0) = -kt/m ⇒ v(t) = v_0 e^(-kt/m).

- **Centripetal Force Rule:** Centripetal force F_c is NEVER drawn as a separate arrow on a free-body diagram; it is the net radial sum of actual forces (tension, gravity, normal force, or friction).

EXAMPLE 2.4 **Solving the Vertical Drag Differential Equation by Separation of Variables**

> **Problem:** A sphere of mass m is released from rest at t = 0 in a fluid that exerts a resistive drag force F_drag = -kv opposing velocity. Starting from Newton's Second Law, write the differential equation for v(t) and solve it by separation of variables to find v(t) and a(t).

STEP-BY-STEP SOLUTION:

1. Step 1 (Differential Equation): Taking downward as positive, ∑F_y = mg - kv = m(dv/dt), or dv/dt = (g - (k/m)v) = -(k/m)(v - mg/k).
2. Step 2 (Separate Variables): Move all v terms to the left and all t terms to the right: dv / (v - mg/k) = -(k/m) dt.
3. Step 3 (Integrate with Limits): Integrate from v' = 0 at t' = 0 to v at t: ∫(0 to v) dv' / (v' - mg/k) = ∫(0 to t) -(k/m) dt' ⇒ ln[(v - mg/k) / (-mg/k)] = -kt/m.
4. Step 4 (Exponentiate & Solve): 1 - (kv)/(mg) = e^(-kt/m) ⇒ v(t) = (mg/k)(1 - e^(-kt/m)) = v_T(1 - e^(-kt/m)). Differentiating v(t) gives a(t) = dv/dt = g e^(-kt/m).

Final Answer: v(t) = (mg/k)(1 - e^(-kt/m)) and a(t) = g e^(-kt/m).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: When separating variables in m(dv/dt) = mg - kv, you MUST keep the entire binomial (mg - kv) together in the denominator as dv/(mg - kv)—never split mg and kv across the equals sign before integrating!

✅ **CHAPTER 2 MASTERY CHECKLIST (AP EXAM READINESS)**

☑ Evaluate continuous center of mass x_cm = (1/M) ∫ x dm by substituting dm = λ(x) dx and integrating both numerator and denominator.

☑ Draw clean FBDs containing only real physical forces originating on the object, and apply Newton's Shell Theorem for r < R inside non-uniform spheres.

☑ Separate variables and integrate first-order drag ODEs m(dv/dt) = -kv and m(dv/dt) = mg - kv with explicit initial-to-final limits.

☑ Analyze horizontal, vertical, and banked circular motion by equating net inward radial force ∑F_r to mv^2/r.

CHAPTER 3 · CORE CURRICULUM

AP Exam Weighting: 14–17%

Unit 3: Work, Energy & Power

Energy methods provide a powerful scalar alternative to vector dynamics whenever forces depend on position. This unit develops kinetic energy, the dot-product and calculus integral definitions of work, the Work-Energy Theorem, gravitational and elastic potential energy derivations, extracting conservative forces from potential energy curves via F_x = -dU/dx, universal gravitational potential energy U_g = -GMm/r, mechanical energy conservation with thermal friction losses, vertical loop-the-loop synthesis, and instantaneous/average power.

CHAPTER 3 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$K = \frac{1}{2}mv^2$ | $W_{net} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$	Translational kinetic energy and the Work-Energy Theorem
$W = \int_a^b \vec{F} \cdot d\vec{r} = \int_{x_1}^{x_2} F_x(x)\,dx$ | $W = Fd\cos\theta$	Work done by a variable force (signed area under F_x vs. x) and constant force
$\Delta U = -W_{cons} = -\int F_x(x)\,dx$ | $F_x = -dU/dx$	Potential energy change and conservative force as negative derivative of U(x)
$\Delta U_g = mg\Delta y$ | $U_s = \frac{1}{2}kx^2$ | $U_g(r) = -G m_1 m_2 / r$	Near-Earth gravitational, ideal spring elastic, and universal gravitational potential energies
$K_i + U_i = K_f + U_f + \Delta E_{th}$ (where $\Delta E_{th} = f_k d$)	Conservation of energy including non-conservative thermal energy dissipation
$P_{avg} = \Delta E / \Delta t = W / \Delta t$ | $P = dW/dt = \vec{F} \cdot \vec{v} = Fv\cos\theta$	Average and instantaneous mechanical power in watts (1 W = 1 J/s)

§ 3.1 Mechanical Energy, Kinetic Energy & Dot-Product / Integral Work

In mechanics, **mechanical energy** is ordered energy—either stored in a system's configuration (gravitational U_g or elastic U_s) or carried in macroscopic motion (kinetic energy K)—that can directly perform mechanical work, such as spinning a turbine. Disordered microscopic **thermal energy** (warmth from rubbing hands) or dispersed sound waves cannot be spontaneously recaptured to do mechanical work. In any closed, isolated system, total energy is strictly conserved ($\Delta E_{total} = 0$).

Kinetic energy is the scalar energy of motion: **$K = \frac{1}{2}mv^2$**, measured in **joules ($1\ J = 1\ N{\cdot}m = 1\ kg{\cdot}m^2/s^2$)**. Because speed is squared, K is always a non-negative scalar independent of direction, and grows quadratically with speed: accelerating a mass from 10 m/s to 12 m/s ($12^2 - 10^2 = 44$) requires nearly four times as much work as accelerating the same mass from 2 m/s to 4 m/s ($4^2 - 2^2 = 12$).

Work (W) is the mechanical transfer of energy into or out of a system by a force acting through a displacement: **$W = \int_a^b \vec{F} \cdot d\vec{r}$**. The vector dot product selects only the force component parallel to displacement, reducing for a constant force over displacement d to **$W = Fd\cos\theta$**. When force varies with position F_x(x), work is evaluated via the definite integral **$W = \int_{x_1}^{x_2} F_x(x)\,dx$**, which geometrically equals the **signed area** under the F_x vs. x curve.

- **Zero Displacement (d = 0):** Holding a heavy rock stationary does W = 0 J of mechanical work regardless of muscle fatigue.
- **Acute Angle ($0° \le \theta < 90°$):** $\cos\theta > 0 \Rightarrow$ positive work (W > 0), adding energy to the system (e.g., gravity pulling a falling rock downward).
- **Perpendicular Force ($\theta = 90°$):** $\cos 90° = 0 \Rightarrow$ zero work (W = 0); normal force on a sliding block and inward centripetal force in circular motion never do work.

- **Obtuse Angle (90° < θ ≤ 180°):** cos θ < 0 ⇒ negative work (W < 0), removing mechanical energy from the system (e.g., kinetic friction at θ = 180°).

EXAMPLE 3.1 **Calculus Proof of the Work-Energy Theorem (W_net = ΔK)**

Problem: Using Newton's Second Law F_net = m(dv/dt) and the chain rule, prove from the work integral W_net = ∫(x₀ to x_f) F_net dx that the net work done on a particle equals its change in kinetic energy ΔK = ½mv_f² - ½mv₀².

STEP-BY-STEP SOLUTION:

1. Step 1: Substitute Newton's Second Law F_net = m(dv/dt) into the work integral: W_net = ∫(x₀ to x_f) m (dv/dt) dx.
2. Step 2: Apply the chain rule to swap differentials: (dv/dt) dx = dv (dx/dt) = v dv, changing the integration variable from position x to speed v.
3. Step 3: Integrate with respect to v from initial speed v₀ to final speed v_f: W_net = ∫(v₀ to v_f) m v dv = [½mv²](v₀ to v_f) = ½mv_f² - ½mv₀² = ΔK.
4. Step 4: Conversely, differentiating with respect to x shows that the spatial rate of change of kinetic energy equals net force: dK/dx = F_net.

Final Answer: W_net = ½mv_f² - ½mv₀² = ΔK (and dK/dx = F_net).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Always check the horizontal axis of a Force graph! The area under a Force vs. Position graph F(x) is Work (J = ΔK), whereas the area under a Force vs. Time graph F(t) is Impulse (N·s = Δp).

§ 3.2 Potential Energy Derivations & Reading Potential Energy Curves U(x)

Potential energy (U) is energy stored in a system due to the relative positions of its interacting parts, defined by the negative of the work done by an internal conservative force: **ΔU = -W_cons = -∫ F_x(x) dx**. Near Earth's surface, as an object rises from y = a to y = b, downward gravity mg acts at 180° to upward dy (cos 180° = -1), doing work W_g = ∫(a to b) -mg dy = -mg Δy; thus **ΔU_g = -W_g = +mg Δy**. Similarly, compressing or stretching an ideal spring (F_s = -kx) from 0 to x stores elastic potential energy **U_s = -∫(0 to x) (-kx) dx = ∫(0 to x) kx dx = ½kx²**. Spring energy grows quadratically because the restoring force kx grows larger with every additional millimeter of stretch.

Differentiating ΔU = -∫ F_x dx with respect to x yields one of the most important equations in AP Physics C: **F_x = -dU/dx**. A conservative force always equals the **negative slope** of its potential energy vs. position graph, meaning the force always pushes the particle downhill toward lower potential energy. Differentiating U_g = mgy gives F_y = -d(mgy)/dy = -mg, and differentiating U_s = ½kx² gives F_x = -d(½kx²)/dx = -kx.

On any potential energy curve U(x) with horizontal total mechanical energy line E_mech: (1) kinetic energy at position x is the vertical gap **K(x) = E_mech - U(x)**; (2) points where E_mech intersects U(x) have K = 0 and represent **turning points** beyond which the particle cannot travel; and (3) points where slope dU/dx = 0 have zero force (F_x = 0) and represent **equilibrium points**.

- **Stable Equilibrium (Valley / Local Minimum):** dU/dx = 0 and d²U/dx² > 0 (concave up); a small displacement creates a restoring force pushing the particle back toward the minimum.
- **Unstable Equilibrium (Hill Peak / Local Maximum):** dU/dx = 0 and d²U/dx² < 0 (concave down); a small nudge creates a force pushing the particle further away from the peak.
- **Neutral Equilibrium (Flat Plateau):** dU/dx = 0 and d²U/dx² = 0 (horizontal line); displacing the particle leaves it in equilibrium with zero force.

Table 3.1 · Interpreting Potential Energy Landscapes U(x)

Feature on U(x) Graph	Mathematical Condition	Physical Meaning for the Particle
Rising Slope (dU/dx > 0)	$F_x = -dU/dx < 0$	Force pushes particle in the -x direction (leftward)
Falling Slope (dU/dx < 0)	$F_x = -dU/dx > 0$	Force pushes particle in the +x direction (rightward)
Local Minimum (Bottom of Valley)	$dU/dx = 0$, $d^2U/dx^2 > 0$	Stable Equilibrium (maximum K, small oscillations are SHM)
Local Maximum (Peak of Hill)	$dU/dx = 0$, $d^2U/dx^2 < 0$	Unstable Equilibrium (slight perturbation causes runaway motion)
Intersection U(x) = E_mech	$K = E_{mech} - U(x) = 0$	Turning Point (particle momentarily stops and reverses direction)

EXAMPLE 3.2 **Deriving Universal Gravitational Potential Energy U_g(r) = -GMm/r**

Problem: For a satellite of mass m moving from infinite separation ($r = \infty$, where $U_g(\infty) = 0$) to radial distance r from a planet of mass M, integrate Newton's Law of Universal Gravitation to derive $U_g(r)$ and explain the physical meaning of the negative sign.

STEP-BY-STEP SOLUTION:

1. Step 1: Write the radial gravitational force: $F_r(r') = -G M m / r'^2$ (attractive, pointing inward in the $-\hat{r}$ direction).
2. Step 2: Apply $U_g(r) - U_g(\infty) = -\int(\infty \text{ to } r) F_r(r')\, dr' = -\int(\infty \text{ to } r) (-GMm\, r'^{-2})\, dr' = GMm \int(\infty \text{ to } r) r'^{-2}\, dr'$.
3. Step 3: Evaluate the antiderivative $[-r'^{-1}](\infty \text{ to } r) = (-1/r) - (0) = -1/r$, yielding $U_g(r) = -GMm / r$.
4. Step 4: Because $U_g(\infty) = 0$ by convention and gravity is purely attractive, potential energy decreases below zero as masses fall closer together.

Final Answer: $U_g(r) = -GMm / r$ (strictly negative at all finite distances r, approaching 0 as $r \to \infty$).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never drop the minus sign in $F_x = -dU/dx$ or in $U_g = -GMm/r$! As a satellite moves to a higher orbit (larger r), $U_g = -GMm/r$ becomes LESS negative, which means gravitational potential energy INCREASES.

§ 3.3 Conservative vs. Non-Conservative Forces & Loop-the-Loop Synthesis

Forces are classified into **conservative** and **non-conservative** forces. A force is **conservative** (gravity, ideal spring) if the work it does depends solely on the initial and final positions—completely independent of the path taken—and the work done around any closed loop is zero ($\oint \vec{F} \cdot d\vec{r} = 0$). Conservative forces reversibly convert energy between kinetic and potential forms. By contrast, kinetic friction and air drag are **non-conservative forces** whose work depends on the total path length d. When a block slides distance d across a rough surface, kinetic friction opposes velocity at 180°, doing negative work **$W_f = -f_k d$** and converting mechanical energy irreversibly into positive thermal energy: **$\Delta E_{thermal} = +f_k d$**.

Writing the full conservation of energy equation between initial and final states yields: **$K_i + U_{gi} + U_{si} = K_f + U_{gf} + U_{sf} + f_k d$**. For a block of mass m released from rest at height h on a frictionless ramp that compresses a horizontal bottom spring of stiffness k, energy transforms from $U_{gi} = mgh$ at the top to $K = \frac{1}{2}mv^2$ on the flat, and finally to $U_{sf} = \frac{1}{2}kx_{max}^2$ when the block momentarily stops at maximum compression, yielding **$x_{max} = \sqrt{(2mgh / k)}$**.

A classic AP Physics C synthesis problem combines conservation of mechanical energy with centripetal dynamics on a vertical **loop-the-loop track** of radius R. At the inside top of the loop (height $y = 2R$), both downward gravity mg and downward normal force F_N point toward the center of the circle: **$F_N + mg = mv_{top}^2 / R$**. The minimum speed to barely maintain contact at the top occurs when normal force drops to zero ($F_N = 0$), giving **$v_{top,min} = \sqrt{(gR)}$**.

- **Minimum Release Height for a Vertical Loop:** Setting $mgh_{min} = mg(2R) + \frac{1}{2}m\, v_{top,min}^2 = 2mgR + \frac{1}{2}m(gR) = 2.5mgR$ cancels m and g to give $h_{min} = 2.5R = (5/2)R$.

EXAMPLE 3.3 **Minimum Release Height to Clear a Frictionless Vertical Loop-the-Loop**

> **Problem:** A small block of mass m is released from rest at height h on a frictionless incline that feeds into a vertical circular loop of radius R. Determine the minimum release height h (in terms of R) so that the block never loses contact with the track at the top of the loop.

STEP-BY-STEP SOLUTION:

1. Step 1 (Centripetal Force at Loop Top): At the top of the loop (y = 2R), both F_N and mg point downward toward the center: $\sum F_r = F_N + mg = mv_{top}^2 / R$.
2. Step 2 (Threshold of Losing Contact): Set F_N = 0 for minimum speed at the top: $mg = mv_{top,min}^2 / R \Rightarrow v_{top,min}^2 = gR$.
3. Step 3 (Conservation of Mechanical Energy): Equate initial energy at release (y = h, v = 0) to energy at the top of the loop (y = 2R, v = v_top,min): $mgh_{min} = mg(2R) + \frac{1}{2}m v_{top,min}^2$.
4. Step 4 (Solve for h_min): Substitute $v_{top,min}^2 = gR$: $mgh_{min} = 2mgR + \frac{1}{2}mgR = 2.5mgR \Rightarrow h_{min} = 2.5R = (5/2)R$.

Final Answer: h_min = 2.5R (or 5R/2).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: A block released from h = 2R (the exact height of the top of the loop) will NOT make it around the loop! At y = 2R it would need $v_{top} = \sqrt{gR} > 0$ to stay on the circular track, requiring an extra ½R of initial height (h_min = 2.5R).

§ 3.4 Mechanical Power: Average, Instantaneous & Dot-Product Form

While work and energy measure the total quantity of joules transferred, **power (P)** measures the *time rate* at which energy is transferred or work is performed. **Average power** over time interval Δt is **$P_{avg} = \Delta E/\Delta t = W/\Delta t$**, and **instantaneous power** is the time derivative **$P = dE/dt = dW/dt$**.

Substituting the differential work $dW = \vec{F} \cdot d\vec{r}$ into $P = dW/dt$ yields **$P = \vec{F} \cdot (d\vec{r}/dt) = \vec{F} \cdot \vec{v} = Fv\cos\theta$**. The SI unit of power is the **watt (W)**, where **$1\ W = 1\ J/s = 1\ kg \cdot m^2/s^3$** (and 1 horsepower ≈ 746 W).

- **Work vs. Power Contrast:** Climbing a staircase of height h requires the exact same mechanical work against gravity (W = mgh) whether you sprint or walk, but sprinting in a shorter time Δt requires a much higher power output P = W/Δt.
- **Constant-Power Acceleration:** If an engine delivers constant power P_0 to a car of mass m (with no drag), $P_0 = Fv = m(dv/dt)v \Rightarrow \int v\,dv = \int (P_0/m)\,dt \Rightarrow \frac{1}{2}v^2 = (P_0/m)t \Rightarrow v(t) \propto \sqrt{t}$.

EXAMPLE 3.4 **Staircase Climb: Contrasting Total Work Done vs. Power Output**

> **Problem:** A 60 kg athlete starts at y = 0 m and climbs to the top of a 100 m high stadium staircase (use g = 10 m/s²). Compare the mechanical work done against gravity and the average power output if the athlete (a) sprints up in 100 s versus (b) walks up in 10 minutes (600 s).

STEP-BY-STEP SOLUTION:

1. Step 1 (Mechanical Work / Potential Energy Gain): In both cases, $\Delta U_g = mg\,\Delta y = (60\ kg)(10\ m/s^2)(100\ m) = 60{,}000\ J = 60\ kJ$.
2. Step 2 (Sprinting Power in 100 s): $P_{sprint} = W / \Delta t_1 = 60{,}000\ J / 100\ s = 600\ W$.
3. Step 3 (Walking Power in 600 s): $P_{walk} = W / \Delta t_2 = 60{,}000\ J / 600\ s = 100\ W$.

Final Answer: Work is 60,000 J in both cases; Power is 600 W sprinting vs. 100 W walking.

⚠️ AP EXAM TRAP & FRQ SCORING GUIDELINE

AP Exam Trap: Do not confuse the italic symbol W for Work (measured in joules, J) with the unit abbreviation W for watts (joules per second, J/s)!

✅ CHAPTER 3 MASTERY CHECKLIST (AP EXAM READINESS)

☑ Evaluate variable-force work $W = \int F_x(x)\,dx$ analytically and from the signed area under an F_x vs. x graph.

☑ Extract force $F_x = -dU/dx$ from any $U(x)$ function or graph, and classify local minima as stable equilibria and local maxima as unstable equilibria.

☑ Apply energy conservation $K_i + U_i = K_f + U_f + f_k d$ across ramps, springs, and vertical circular loops ($v_{top,min} = \sqrt{(gR)}$, $h_{min} = 2.5R$).

☑ Calculate instantaneous power via $P = dW/dt = \vec{F} \cdot \vec{v} = Fv\cos\theta$.

CHAPTER 4 · CORE CURRICULUM

AP Exam Weighting: 12–18%

Unit 4: Systems of Particles & Linear Momentum

Linear momentum and impulse govern collisions, explosions, and multi-particle interactions where large internal forces act over brief time intervals. This unit covers linear momentum $\vec{p} = m\vec{v}$, Newton's Second Law in derivative form $\vec{F}_{net} = d\vec{p}/dt$, the Impulse-Momentum Theorem $\vec{J} = \int \vec{F}(t)\,dt = \Delta\vec{p}$, center-of-mass velocity invariance, 1D elastic/inelastic collisions and explosions, 2D glancing collisions, and the two-stage ballistic pendulum.

CHAPTER 4 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$\vec{p} = m\vec{v} \mid K = p^2/(2m)$	Linear momentum vector and kinetic energy in terms of momentum magnitude
$\sum\vec{F} = d\vec{p}/dt = d(m\vec{v})/dt$	Newton's Second Law in general momentum-derivative form
$\vec{J} = \int (t_1 \text{ to } t_2)\ \vec{F}(t)\,dt = \vec{F}_{avg}\,\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_0$	Impulse-Momentum Theorem (signed area under Force vs. Time curve)
$\vec{p}_{total} = \sum m_i\vec{v}_i = M_{total}\vec{v}_{cm}$	System momentum and center-of-mass velocity relation
$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \mid v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$	Conservation of linear momentum and 1D elastic relative-velocity shortcut
$v_0 = [(m + M)/m]\sqrt{(2gh)}$	Initial bullet speed in a two-stage ballistic pendulum

§ 4.1 Linear Momentum, F_net = dp/dt & The Impulse-Momentum Theorem

Linear momentum ($\vec{p} = m\vec{v}$) is a vector quantity measuring how difficult it is to stop or deflect a moving object, with SI units of **kg·m/s (or N·s)**. Unlike kinetic energy $K = \frac{1}{2}mv^2$ (a non-negative scalar), momentum is a vector pointing in the exact direction of velocity. Expressing kinetic energy in terms of momentum magnitude yields the high-yield relation **$K = p^2/(2m)$**: if two objects carry identical momentum magnitude p, the *lighter* mass m possesses the larger kinetic energy.

Isaac Newton originally formulated his Second Law not as F = ma, but as the time derivative of momentum: **$\sum\vec{F} = d\vec{p}/dt = d(m\vec{v})/dt$**. When mass m is constant, m factors out to give $m(d\vec{v}/dt) = m\vec{a}$; however, $\sum\vec{F} = d\vec{p}/dt$ is more general because it also governs variable-mass systems such as rockets expelling exhaust gas at rate dm/dt. Graphically, the instantaneous slope of a momentum-time graph p(t) equals the net force F_net(t).

Multiplying $\sum\vec{F} = d\vec{p}/dt$ by dt and integrating from t_1 to t_2 produces the **Impulse-Momentum Theorem**: **$\vec{J} = \int (t_1 \text{ to } t_2)\ \vec{F}(t)\,dt = \vec{F}_{avg}\,\Delta t = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_0$**. Geometrically, impulse $\vec{J}$ equals the **signed area under the Force vs. Time curve F(t)**. In any crash from speed v_0 to rest, $\Delta p = -mv_0$ is fixed; airbags, crumple zones, and bending your knees upon landing extend the collision duration Δt by a factor of 10, reducing the average impact force $F_{avg} = \Delta p/\Delta t$ by a factor of 10 (same area under F(t), much lower peak force).

- **Work vs. Impulse Contrast:** Work $W = \int \vec{F} \cdot d\vec{r}$ is a spatial dot-product integral yielding a scalar ($J = \Delta K$); Impulse $\vec{J} = \int \vec{F}(t)\,dt$ is a time integral yielding a vector ($N\cdot s = \Delta\vec{p}$).
- **1D Wall Rebound Sign Rule:** If a ball of mass m moving right at $+v_0$ hits a wall and sticks ($v_f = 0$), $\Delta p = 0 - mv_0 = -mv_0$. If it rebounds elastically left at $-v_0$, then $\Delta p = (-mv_0) - (+mv_0) = -2mv_0$—bouncing transfers twice the impulse and twice the average force of sticking!

EXAMPLE 4.1 **Comparing Impulse for Sticking Clay vs. Rebounding Rubber Ball**

> **Problem:** A 0.50 kg ball traveling horizontally to the right at +10 m/s strikes a vertical wall. Calculate the impulse J_x delivered by the wall to the ball if (a) the ball is a lump of clay that sticks to the wall and stops, versus (b) the ball is a rubber ball that rebounds horizontally to the left at 10 m/s.

STEP-BY-STEP SOLUTION:

1. Step 1: Define rightward as +x. Initial momentum in both cases is p_i = m v_i = (0.50 kg)(+10 m/s) = +5.0 kg·m/s.
2. Step 2 (Case a — Sticking): Final momentum is p_f = 0. Thus J_a = Δp = p_f - p_i = 0 - 5.0 = -5.0 N·s (5.0 N·s to the left).
3. Step 3 (Case b — Elastic Rebound): Final velocity is v_f = -10 m/s, so p_f = (0.50)(-10) = -5.0 kg·m/s. Thus J_b = p_f - p_i = (-5.0) - (+5.0) = -10.0 N·s (10.0 N·s to the left).

Final Answer: (a) J_a = -5.0 N·s; (b) J_b = -10.0 N·s (rebounding requires twice the impulse of stopping).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: When an object rebounds from a wall with the same speed v_0, Δp is NOT zero! Because velocity reverses sign from $+v_0$ to $-v_0$, the magnitude of momentum change is $|\Delta p| = 2mv_0$.

§ 4.2 Conservation of Momentum, System Boundaries & Center-of-Mass Velocity

When two or more objects interact inside a **closed, isolated system** where the net external force is zero (∑F⃗_ext = 0), Newton's Third Law guarantees that internal forces are equal and opposite at every instant (F⃗_1 on 2 = -F⃗_2 on 1). Their internal impulses cancel completely over any collision interval Δt, so the **total linear momentum vector of the system remains strictly constant: ∑p⃗_initial = ∑p⃗_final**. For instance, if a 0.1 kg black marker moving right at +2 m/s (p_1 = +0.2 kg·m/s) collides head-on with an identical 0.1 kg red marker moving left at -2 m/s (p_2 = -0.2 kg·m/s) and they stick together, their total initial momentum is +0.2 + (-0.2) = 0, so they stop dead with final momentum 0.

Whether momentum is conserved depends on how we define our **system boundary**. When a thrown marker hits a stationary wall and stops, the marker alone is *not* an isolated system because the wall's normal force acts as an external force. However, if we expand our system boundary to include the **marker + wall + Earth**, total momentum is 100% conserved: the marker transfers all of its forward momentum $+mv_0$ to the Earth (whose huge mass M_Earth ≈ 6 × 10^{24} kg makes the resulting recoil speed $\Delta v = mv_0/M_{Earth}$ imperceptibly tiny).

Differentiating x_cm with respect to time gives the velocity of the center of mass: **v⃗_cm = (∑ m_i v⃗_i) / M_total = p⃗_total / M_total**, or **p⃗_total = M_total v⃗_cm**. Because p⃗_total is constant whenever ∑F⃗_ext = 0, **the velocity of the center of mass v⃗_cm never changes during an internal collision or explosion!**

- **Invariance of v_cm:** Whether two colliding spheres bounce elastically, stick together inelastically, or deflect in 2D, v⃗_cm after the collision is identical to v⃗_cm before the collision (provided ∑F⃗_ext = 0).

EXAMPLE 4.2 **Center-of-Mass Velocity Before and After a Collision**

Problem: A sphere of mass 2m moves to the right at +5.0 m/s toward a stationary sphere of mass m ($v_2 = 0$) on a frictionless horizontal track. Find the velocity of the system's center of mass (a) before the collision and (b) after the collision.

STEP-BY-STEP SOLUTION:

1. Step 1: Compute v_cm before the collision using $v_{cm} = (m_1v_1 + m_2v_2) / (m_1 + m_2)$: $v_{cm} = [(2m)(5.0) + (m)(0)] / (2m + m) = 10m / (3m) = 10/3$ m/s ≈ +3.33 m/s to the right.
2. Step 2: Because the track is horizontal and frictionless, net external horizontal force is zero ($\sum F_{ext,x} = 0$).
3. Step 3: Therefore, total momentum p_total and center-of-mass velocity v_cm remain unchanged regardless of whether the collision is elastic or inelastic.

Final Answer: v_cm = +10/3 m/s (+3.33 m/s to the right) both before and after the collision.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: If an exam question gives initial masses and velocities and asks for the velocity of the center of mass AFTER a complex elastic or inelastic collision, do NOT waste time solving for the individual final velocities—v_cm after impact equals v_cm before impact!

§ 4.3 1D Collisions (Elastic, Inelastic, Perfectly Inelastic) & Explosions

In every collision inside an isolated system, **total linear momentum is always conserved**. Collisions are classified into three types based on whether mechanical kinetic energy K is conserved: (1) **Perfectly Elastic Collision**: objects rebound with zero permanent deformation or heat loss, so *both* momentum and kinetic energy are conserved ($K_i = K_f$); (2) **Inelastic Collision**: objects bounce apart, but some kinetic energy is lost to thermal energy, sound, or deformation ($K_f < K_i$); and (3) **Perfectly Inelastic Collision**: the colliding objects *stick together* and move with a single common final velocity $\mathbf{v_f = v_{cm} = (m_1v_1i + m_2v_2i)/(m_1 + m_2)}$, losing the maximum possible kinetic energy consistent with momentum conservation.

In a 1D perfectly elastic collision, solving the quadratic kinetic energy equation $\frac{1}{2}m_1v_1i^2 + \frac{1}{2}m_2v_2i^2 = \frac{1}{2}m_1v_1f^2 + \frac{1}{2}m_2v_2f^2$ simultaneously with momentum conservation yields a fast linear shortcut: **the relative speed of approach equals the relative speed of separation: $\mathbf{v_1i - v_2i = -(v_1f - v_2f)}$**. Furthermore, when two objects of **equal mass ($\mathbf{m_1 = m_2}$)** collide elastically in 1D, they simply **swap velocities** ($v_1f = v_2i$ and $v_2f = v_1i$).

An **explosion (or recoil)** is the time-reverse of a perfectly inelastic collision: a single stationary object ($p_i = 0$) breaks apart into two fragments via internal potential energy, so $\mathbf{0 = m_1v_1f + m_2v_2f \Rightarrow m_1|v_1f| = m_2|v_2f|}$. Because both fragments acquire equal momentum magnitude p and $K = p^2/(2m)$, the lighter fragment always receives the larger speed and the larger share of the released kinetic energy.

- **Equal-Mass Billiard Swap:** When a moving cue ball of mass m hits a stationary target ball of equal mass m head-on elastically, the cue ball stops dead ($v_1f = 0$) and the target ball moves forward at the cue ball's exact initial velocity ($v_2f = v_1i$).

Table 4.1 · Complete Classification of Mechanical Interactions in an Isolated System

Interaction Type	Total Momentum $\sum p$	Total Kinetic Energy $\sum K$	Defining Kinematic / Energy Signature
Perfectly Elastic Collision	Conserved	Conserved ($K_f = K_i$)	Objects bounce; $v_1i - v_2i = -(v_1f - v_2f)$
Partially Inelastic Collision	Conserved	Decreases ($K_f < K_i$)	Objects bounce apart with thermal/sound loss
Perfectly Inelastic Collision	Conserved	Maximum Loss ($K_f < K_i$)	Objects stick together: $v_1f = v_2f = v_{cm}$
Explosion / Recoil	Conserved	Increases ($K_f > K_i$)	Stored U converts to K; $p_1f = -p_2f$ (from rest)

EXAMPLE 4.3 **1D Perfectly Inelastic Coupling & Kinetic Energy Loss**

> **Problem:** A 2.0 kg cart moving right at 6.0 m/s collides with and couples to a stationary 4.0 kg cart on a frictionless track. Calculate (a) the final speed of the coupled carts and (b) the mechanical kinetic energy lost to heat and sound.

STEP-BY-STEP SOLUTION:

1. Step 1 (Momentum Conservation): $m_1v_1i + m_2v_2i = (m_1 + m_2)v_f \Rightarrow (2.0)(6.0) + (4.0)(0) = (2.0 + 4.0)v_f \Rightarrow 12 = 6.0\ v_f \Rightarrow v_f = 2.0$ m/s.
2. Step 2 (Initial Kinetic Energy): $K_i = \frac{1}{2}m_1v_1i^2 = \frac{1}{2}(2.0)(6.0)^2 = 36$ J.
3. Step 3 (Final Kinetic Energy & Loss): $K_f = \frac{1}{2}(m_1 + m_2)v_f^2 = \frac{1}{2}(6.0)(2.0)^2 = 12$ J. Energy lost = $K_i - K_f = 36$ J - 12 J = 24 J (two-thirds of the initial kinetic energy is dissipated).

Final Answer: $v_f = 2.0$ m/s; $\Delta K_{lost} = 24$ J.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never assume kinetic energy is conserved in a collision unless the problem explicitly states the collision is 'elastic' or provides rebounding speeds that satisfy $K_i = K_f$!

§ 4.4 2D Glancing Collisions & The Two-Stage Ballistic Pendulum

Because linear momentum is a vector, in a **2D glancing collision** momentum is conserved independently along the x-axis ($\sum p_{ix} = \sum p_{fx}$) and along the y-axis ($\sum p_{iy} = \sum p_{fy}$). If ball 1 initially moves purely along the +x-axis and strikes stationary ball 2 so that ball 1 deflects downward at angle θ_1 and ball 2 deflects upward at angle θ_2, the total initial y-momentum is strictly zero ($p_{iy} = 0$). Therefore, after impact the upward y-momentum of ball 2 must equal the downward y-momentum of ball 1 in magnitude: $\mathbf{p_1f \sin \theta_1 = p_2f \sin \theta_2}$, while along the x-axis $\mathbf{p_1i = p_1f \cos \theta_1 + p_2f \cos \theta_2}$. Writing the y-momentum equation first is always the fastest way to relate p_1f and p_2f.

The **ballistic pendulum**—where a bullet of mass m and initial speed v_0 embeds into a stationary hanging wooden block of mass M, and the combined mass (m + M) swings upward by vertical height h—must be solved in **two distinct stages**: (Stage 1: Rapid Embedding Collision) horizontal momentum is conserved ($\mathbf{mv_0 = (m + M)v_{after}}$), whereas mechanical energy is *not* conserved due to wood friction; (Stage 2: Upward Pendulum Swing) external gravity acts so horizontal momentum is no longer conserved, but mechanical energy *is* conserved: $\mathbf{\frac{1}{2}(m + M)v_{after}^2 = (m + M)gh \Rightarrow v_{after} = \sqrt{(2gh)}}$. Combining the two stages yields $\mathbf{v_0 = [(m + M)/m]\sqrt{(2gh)}}$.

- **Stage Separation Rule:** Never equate the bullet's initial kinetic energy $\frac{1}{2}mv_0^2$ directly to the block-bullet system's final potential energy (m + M)gh—almost all of the bullet's initial kinetic energy (fraction M/(m+M)) is lost as heat during the embedding collision!

EXAMPLE 4.4 Complete 2D Glancing Collision Analysis

Problem: Ball 1 of mass 2m moves right along the +x-axis at 5.0 m/s and strikes a stationary ball 2 of mass m. After the collision, ball 1 deflects downward at $\theta_1 = 20°$ below the +x-axis and ball 2 deflects upward at $\theta_2 = 40°$ above the +x-axis. Find the final speed v_1f of ball 1.

STEP-BY-STEP SOLUTION:

1. Step 1 (Initial Momentum): p_ix = (2m)(5.0) = 10m; p_iy = 0.
2. Step 2 (y-Momentum Conservation): Since p_iy = 0, downward y-momentum of ball 1 equals upward y-momentum of ball 2: $p_1f \sin 20° = p_2f \sin 40° \Rightarrow 0.342\ p_1f = 0.643\ p_2f \Rightarrow p_1f = 1.88\ p_2f$.
3. Step 3 (x-Momentum Conservation): $p_1f \cos 20° + p_2f \cos 40° = 10m \Rightarrow 0.940\ p_1f + 0.766\ p_2f = 10m$.
4. Step 4 (Substitute & Solve): Substitute $p_1f = 1.88\ p_2f$: $0.940(1.88\ p_2f) + 0.766\ p_2f = 2.53\ p_2f = 10m \Rightarrow p_2f = 3.95m$ and $p_1f = 7.42m$. Dividing p_1f by mass 2m yields $v_1f = 7.42m / (2m) = 3.71$ m/s.

Final Answer: $p_2f = 3.95m$, $p_1f = 7.42m$, and $v_1f = 3.7$ m/s.

⚠ AP EXAM TRAP & FRQ SCORING GUIDELINE

AP Exam Trap: Setting $\frac{1}{2}mv_0^2 = (m + M)gh$ in a ballistic pendulum problem earns zero credit for the collision stage! Always use momentum conservation during the collision and energy conservation only after the bullet is embedded.

✅ CHAPTER 4 MASTERY CHECKLIST (AP EXAM READINESS)

☑ Relate kinetic energy and momentum via $K = p^2/(2m)$ and compute impulse $J\vec{} = \int F\vec{}(t)\,dt = \Delta p\vec{}$ from Force vs. Time graphs.

☑ Apply conservation of momentum $\sum p\vec{}_i = \sum p\vec{}_f$ and recognize that center-of-mass velocity $v\vec{}_{cm}$ is invariant when $\sum F\vec{}_{ext} = 0$.

☑ Use the 1D elastic relative-velocity shortcut $v_1i - v_2i = -(v_1f - v_2f)$ and 2D orthogonal component equations ($p_1f \sin \theta_1 = p_2f \sin \theta_2$).

☑ Split multi-stage problems (like the ballistic pendulum) into a momentum-conserving collision stage followed by an energy-conserving swing stage.

CHAPTER 5 · CORE CURRICULUM

AP Exam Weighting: 10–15%

Unit 5: Torque & Rotational Dynamics

Every linear kinematic and dynamic quantity has a direct rotational analog. This unit establishes angular kinematics (θ, ω, α) and the linear-angular bridge equations, vector torque $\vec{\tau} = \vec{r} \times \vec{F}$ and the Right-Hand Rule, extended free-body diagrams and static equilibrium ($\sum \vec{F} = 0$, $\sum \vec{\tau} = 0$), discrete and continuous rotational inertia $I = \int r^2\, dm$, the Parallel-Axis Theorem $I' = I_{cm} + Md^2$, and Newton's Second Law for rotation $\sum \tau = I\alpha$ applied to massive pulleys and pivoted rods.

CHAPTER 5 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$\omega = d\theta/dt \mid \alpha = d\omega/dt = d^2\theta/dt^2 \mid s = r\theta,\ v_t = r\omega,\ a_t = r\alpha$	Angular calculus kinematics and linear-angular bridge relations (θ in radians)
$a_{total} = \sqrt{(a_t^2 + a_c^2)}$ where $a_c = v^2/r = \omega^2 r$	Total linear acceleration of a point on a rotating rigid body
$\vec{\tau} = \vec{r} \times \vec{F} \mid \lvert\vec{\tau}\rvert = r\,F \sin\theta = r_\perp F$	Torque vector cross product and lever-arm magnitude (N·m)
$\sum F_x = 0, \sum F_y = 0, \sum \tau = 0$ (about any pivot)	Conditions for complete static equilibrium of a rigid body
$I = \sum m_i r_i^2 = \int r^2\, dm \mid I' = I_{cm} + M\,d^2$	Rotational inertia (moment of inertia) and the Parallel-Axis Theorem
$\sum \vec{\tau} = I\,\vec{\alpha} \mid a = (m_2 - m_1)g / (m_1 + m_2 + I/R^2)$	Newton's Second Law for rotation and Atwood acceleration with a massive pulley

§ 5.1 Rotational Kinematics & Linear-Angular Bridge Equations

Every linear variable in Units 1 and 2 maps directly to a rotational analog: linear position x becomes **angular position θ** (measured in **radians**, where 1 rev = 360° = 2π rad), linear velocity v = dx/dt becomes **angular velocity ω = dθ/dt** (in rad/s), linear acceleration a = dv/dt becomes **angular acceleration α = dω/dt = d²θ/dt²** (in rad/s²), mass m becomes **rotational inertia I** (in kg·m²), and force F becomes **torque τ** (in N·m).

Because θ(t), ω(t), and α(t) obey the exact same derivative and integral ladder as x(t), v(t), and a(t), constant angular acceleration α yields the three rotational kinematic equations: (1) **$\omega = \omega_0 + \alpha t$** (lacks Δθ); (2) **$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$** (lacks ω); and (3) **$\omega^2 = \omega_0^2 + 2\alpha\,\Delta\theta$** (lacks t). When α(t) varies with time, we integrate: $\Delta\omega = \int \alpha(t)\, dt$ and $\Delta\theta = \int \omega(t)\, dt$.

To connect the angular motion of a rotating wheel to the linear motion of a point at radial distance r from the axis, we use the definition of a radian: arc length **$s = r\theta$**. Differentiating with respect to time yields tangential speed **$v_t = r\omega$**, and differentiating a second time yields tangential acceleration **$a_t = r\alpha$** (which measures how fast linear speed changes tangent to the circle). Because any point on a spinning wheel also experiences inward radial centripetal acceleration **$a_c = v^2/r = \omega^2 r$** perpendicular to a_t, the total linear acceleration magnitude is **$a_{total} = \sqrt{(a_t^2 + a_c^2)}$**.

- **Radian Mode Rule:** The bridge equations $s = r\theta$, $v_t = r\omega$, and $a_t = r\alpha$ are valid ONLY when θ, ω, and α are expressed in radians (not degrees or revolutions).
- **Rigid Body Definition:** In a rigid body, all particles maintain fixed relative distances and rotate with the exact same angular velocity ω and angular acceleration α (unlike a can of liquid soup that sloshes internally).

Table 5.1 · Complete Dictionary of Linear and Rotational Analogies

Physical Concept	Linear Variable & Unit	Rotational Variable & Unit	Bridge / Defining Relation
Position / Displacement	x, Δx (m)	θ, Δθ (rad)	s = rθ
Velocity	v = dx/dt (m/s)	ω = dθ/dt (rad/s)	v_t = rω
Acceleration	a = dv/dt (m/s²)	α = dω/dt (rad/s²)	a_t = rα (and a_c = ω²r)
Inertia / Resistance	Mass m (kg)	Rotational Inertia I (kg·m²)	I = ∑ m_i r_i² = ∫ r² dm
Cause of Acceleration	Force F⃗ (N)	Torque τ⃗ (N·m)	τ⃗ = r⃗ × F⃗
Newton's Second Law	∑F⃗ = m a⃗	∑τ⃗ = I α⃗	Coupled via a = rα, τ = rF sin θ

EXAMPLE 5.1 **Calculating Torque via Cross Product & Right-Hand Rule**

Problem: A mechanic pulls on a wrench handle at radial distance r = 0.50 m from the bolt axis with a force of F = 20 N at an angle of θ = 70° relative to the outward radius vector r⃗ (turning the wrench clockwise in the plane of the page). Find the magnitude and vector direction of the torque.

STEP-BY-STEP SOLUTION:

1. Step 1: Apply the torque magnitude formula |τ⃗| = |r⃗ × F⃗| = r F sin θ.
2. Step 2: Substitute r = 0.50 m, F = 20 N, and θ = 70°: τ = (0.50 m)(20 N) sin 70° = 10 × 0.9397 = 9.4 N·m.
3. Step 3: Apply the Right-Hand Rule: place r⃗ and F⃗ tail-to-tail, point the fingers of your right hand along r⃗, and curl them through the smaller angle (70°) toward F⃗. Because the rotation is clockwise, your right thumb points into the page (-k̂, traditionally negative).

Final Answer: τ = 9.4 N·m directed into the page (clockwise, or -9.4 N·m).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: In τ = rF sin θ, r is the distance from the pivot axis to the exact point where the force is applied! If you grip a wrench halfway down its handle, r is half the wrench length.

§ 5.2 Extended Free-Body Diagrams & Static Equilibrium (∑F = 0, ∑τ = 0)

When drawing an **extended free-body diagram** for a rigid body on the AP Physics C exam, you **cannot shrink the object to a single dot**. Because torque τ = rF sin θ depends on *where* each force acts, you lose credit unless every force arrow starts at its exact physical point of application: gravity F_g = Mg starts at the center of mass, normal and hinge forces start at the contact surface, and cord tension T starts where the cord attaches.

For a rigid body to remain in complete **static equilibrium**, two conditions must hold simultaneously: (1) **Translational equilibrium: ∑F_x = 0 and ∑F_y = 0**; and (2) **Rotational equilibrium: ∑τ = 0 about any chosen pivot axis**.

When setting up ∑τ = 0, always place your reference pivot axis right at the location of an **unknown force you do not need to find** (such as a wall hinge or support pin). Because any force acting directly at the pivot has lever arm r = 0 (or acts along the line of the pivot with θ = 0° or 180°), its torque about that pivot is identically zero, eliminating the unknown force from your torque equation immediately.

- **Sign Convention for 2D Torques:** Counterclockwise (CCW) torques point out of the page (+k̂) and are traditionally taken as positive (+); clockwise (CW) torques point into the page (-k̂) and are taken as negative (-).

EXAMPLE 5.2 **Static Equilibrium of a Hinged Horizontal Beam**

> **Problem:** A uniform horizontal beam of mass M and length L is attached to a wall by a frictionless pin hinge at its left end and supported at its right end (r = L) by a cable pulling upward at angle θ above the horizontal. Derive an expression for the cable tension T.

STEP-BY-STEP SOLUTION:

1. Step 1: Choose the left wall hinge as the pivot axis (r = 0) so the unknown horizontal and vertical hinge forces (F_hx, F_hy) produce zero torque.
2. Step 2: Gravity Mg acts straight down at the center of mass (r = L/2, angle 90°), producing a clockwise torque τ_g = -Mg(L/2) sin 90° = -½MgL.
3. Step 3: Cable tension T acts at the right end (r = L) at angle θ, producing a counterclockwise torque τ_T = +L T sin θ.
4. Step 4: Set ∑τ_hinge = 0: L T sin θ - ½MgL = 0 ⇒ T = Mg / (2 sin θ).

Final Answer: T = Mg / (2 sin θ).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: A force acting at the pivot has zero torque about that pivot, NOT zero force! After finding T from ∑τ = 0, you must still use ∑F_x = 0 and ∑F_y = 0 to find the non-zero hinge reaction forces.

§ 5.3 Rotational Inertia (I = ∑mr², I = ∫ r² dm) & Parallel-Axis Theorem

Rotational inertia (moment of inertia, I) measures how difficult it is to change an object's angular velocity about a specific axis, with SI units of **kg·m²**. For a discrete system of point masses rotating about a common axis, **I = ∑ I_i = ∑ m_i r_i²**, where r_i is the perpendicular distance from the rotation axis to mass m_i. Because r is squared, two objects with the exact same total mass M and outer radius R have vastly different moments of inertia depending on how far their mass is distributed from the axis: a thin hoop has all its mass at the rim (**I_hoop = MR²**), a uniform solid disk/cylinder distributes mass from 0 to R (**I_disk = ½MR²**), a hollow spherical shell has **I_shell = ⅔MR²**, and a solid sphere concentrates mass near the center (**I_sphere = ⅖MR²**).

For a continuous rigid body, summation becomes integration: **I = ∫ r² dm**, evaluated by substituting **dm = λ(x) dx** (for 1D rods). For a uniform thin rod of mass M and length L, constant linear density is λ = M/L, so dm = (M/L) dx. Integrating about one end (x = 0 to L) gives **I_end = (M/L) ∫(0 to L) x² dx = ⅓ML²**; integrating about the center of mass (x = -L/2 to +L/2) gives **I_cm = (M/L) ∫(-L/2 to L/2) x² dx = (1/12)ML²** (four times easier to spin about its center).

If you already know an object's moment of inertia about its center of mass I_cm, the **Parallel-Axis Theorem** gives its rotational inertia about any parallel axis shifted by distance d: **I' = I_cm + Md²**. Because $Md^2 \geq 0$, an object's rotational inertia is always *smallest* about an axis passing through its center of mass.

- **Parallel-Axis Check on a Uniform Rod:** Shifting from the center of a rod ($I_{cm} = (1/12)ML^2$) to one end (d = L/2) gives $I_{end} = (1/12)ML^2 + M(L/2)^2 = (1/12 + 3/12)ML^2 = ⅓ML^2$.
- **Parallel-Axis Check on a Solid Disk:** Pivoting a solid disk ($I_{cm} = ½MR^2$) about a point on its outer rim (d = R) gives $I_{rim} = ½MR^2 + MR^2 = (3/2)MR^2$.

Table 5.2 · Standard Moments of Inertia for Symmetric Rigid Bodies

Rigid Body & Axis of Rotation	Formula for I	Shape Factor c (in I = cMR² or cML²)
Thin Hoop / Ring about central symmetry axis	$I = M R^2$	c = 1.00
Thin Hollow Spherical Shell about any diameter	$I = (2/3) M R^2$	c = 0.667
Uniform Solid Cylinder or Disk about central axis	$I = (1/2) M R^2$	c = 0.50
Uniform Solid Sphere about any diameter	$I = (2/5) M R^2$	c = 0.40
Uniform Thin Rod about axis through One End (⊥ to rod)	$I = (1/3) M L^2$	c = 0.333
Uniform Thin Rod about axis through Center (⊥ to rod)	$I = (1/12) M L^2$	c = 0.0833

EXAMPLE 5.3 **Discrete 3-Ball Tray & Continuous Non-Uniform Rod Moment of Inertia**

Problem: (a) Three balls rest on a massless rotating tray at center-of-mass distances from the axis: $m_1 = 4.0$ kg at $r_1 = 0.20$ m, $m_2 = 2.0$ kg at $r_2 = 0.40$ m, and $m_3 = 3.0$ kg at $r_3 = 0.30$ m. Find the total rotational inertia I. (b) A non-uniform rod of length R rotates about one end at r = 0 with linear mass density $\lambda(r) = 4r$. Evaluate $I = \int r^2 \, dm$.

STEP-BY-STEP SOLUTION:

1. Step 1 (Part a — Discrete Sum): $I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 = 4.0(0.20)^2 + 2.0(0.40)^2 + 3.0(0.30)^2 = 4(0.04) + 2(0.16) + 3(0.09) = 0.16 + 0.32 + 0.27 = 0.75$ kg·m².
2. Step 2 (Part b — Differential Mass): Since $\lambda(r) = dm/dr = 4r$, we have $dm = 4r\,dr$.
3. Step 3 (Part b — Integrate $r^2\,dm$): $I = \int_0^R r^2(4r\,dr) = \int_0^R 4r^3\,dr = [4r^4/4]_0^R = R^4$.

Final Answer: (a) I = 0.75 kg·m²; (b) $I = R^4$.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: In the Parallel-Axis Theorem $I' = I_{cm} + Md^2$, the first term MUST be I_{cm} about the center of mass! To shift from one non-CM axis to another non-CM axis, first subtract Md_1^2 to get I_{cm}, then add Md_2^2.

§ 5.4 Newton's Second Law for Rotation (∑τ = Iα), Massive Pulleys & Pivoted Rods

Replacing force with net torque ∑τ, mass with rotational inertia I, and linear acceleration with angular acceleration α yields **Newton's Second Law for Rotation: ∑τ = Iα**. When applying ∑τ = Iα, always sum torques about the exact same axis used to compute I, and assign positive signs to torques acting in the direction of α.

When a falling block of mass m unwinds a cord wrapped around a solid disk pulley of mass M, radius R, and rotational inertia $I = \frac{1}{2}MR^2$, we couple translation and rotation: (1) for the falling block (downward +): **mg - T = ma**; (2) for the pulley: **τ = TR = Iα**; and (3) no-slip rope constraint: **α = a/R**. Substituting $I = \frac{1}{2}MR^2$ and α = a/R into TR = Iα cancels R^2 completely, leaving **T = ½Ma** and yielding block acceleration **a = mg / (m + ½M)**.

In an **Atwood machine with a massive pulley** of inertia I and radius R, the cord tensions T_1 and T_2 on the two sides of the pulley are **no longer equal**! For heavier mass m_2 accelerating downward and lighter mass m_1 accelerating upward, we must have **$T_2 > T_1$** so that net torque $(T_2 - T_1)R = I\alpha = I(a/R)$ spins the pulley. Combining $m_2g - T_2 = m_2a$, $T_1 - m_1g = m_1a$, and $T_2 - T_1 = (I/R^2)a$ cancels both tensions to give **$a = (m_2 - m_1)g / (m_1 + m_2 + I/R^2)$**.

- **Effective Pulley Mass:** Notice that a massive pulley simply adds an 'effective rotational mass' I/R^2 (which equals ½M for a solid disk pulley) to the denominator of the system acceleration equation!

EXAMPLE 5.4 Initial Angular & Tip Acceleration of a Pivoted Horizontal Rod

Problem: A uniform rod of mass M, length L, and end rotational inertia $I_{end} = \frac{1}{3}ML^2$ is pivoted frictionlessly at its left end and released from rest in a horizontal position. Find (a) its initial angular acceleration α_0 and (b) the initial downward linear acceleration a_{tip} of its free right tip.

STEP-BY-STEP SOLUTION:

1. Step 1 (Torque about Left Pivot): At the horizontal instant of release, gravity Mg acts straight down at the center of mass at lever arm r = L/2 (angle 90°), producing torque $\tau = Mg(L/2)$.
2. Step 2 (Apply $\sum\tau = I\alpha$): $Mg(L/2) = (\frac{1}{3}ML^2)\,\alpha_0$. Cancel M and one power of L: $\frac{1}{2}g = \frac{1}{3}L\,\alpha_0 \Rightarrow \alpha_0 = 3g / (2L)$.
3. Step 3 (Linear Acceleration of Free Tip at r = L): Using the bridge relation $a_t = r\alpha$ at r = L gives $a_{tip} = L\,\alpha_0 = L(3g / 2L) = (3/2)g = 1.5g$.

Final Answer: (a) $\alpha_0 = 3g / (2L)$; (b) $a_{tip} = 1.5g$ (the free tip initially accelerates downward faster than free-fall g!).

⚠ AP EXAM TRAP & FRQ SCORING GUIDELINE

AP Exam Trap: For a pivoted swinging rod, $\alpha = 3g/(2L)$ applies ONLY at the exact horizontal instant of release! As the rod swings downward through angle θ below the horizontal, the gravitational lever arm shrinks as $(L/2)\cos\theta$, so $\alpha(\theta) = (3g/2L)\cos\theta$ is NOT constant—use energy conservation to find angular speed ω at the bottom!

✅ CHAPTER 5 MASTERY CHECKLIST (AP EXAM READINESS)

- ☑ Connect angular kinematics (θ, ω, α) to linear kinematics via $s = r\theta$, $v_t = r\omega$, $a_t = r\alpha$, and $a_c = \omega^2 r$.
- ☑ Apply $\sum F_x = 0$, $\sum F_y = 0$, and $\sum\tau = 0$ to extended rigid bodies by choosing the pivot at an unknown support force.
- ☑ Derive moments of inertia via $I = \int r^2\,dm$ (with $dm = \lambda\,dx$) and shift axes using $I' = I_{cm} + Md^2$.
- ☑ Couple $\sum F = ma$ and $\sum\tau = I\alpha$ using $a = R\alpha$ for unwinding spools and massive pulleys (where $T_2 \neq T_1$).

CHAPTER 6 · CORE CURRICULUM

AP Exam Weighting: 10–18%

Unit 6: Energy & Momentum of Rotating Systems, Gravitation & Orbits

This unit unites rotational dynamics with mechanical energy conservation, angular momentum conservation, and celestial orbital mechanics. Topics include rotational kinetic energy K_rot = ½Iω², rolling without slipping and the incline shape-factor race, rotational work and power, angular momentum of point particles (L⃗ = r⃗ × p⃗) and rigid bodies (L⃗ = Iω⃗), angular impulse ΔL = ∫ τ dt, rod-ball angular collisions, circular orbital speed, the virial-like orbital energy relations, escape velocity, and Kepler's Three Laws.

CHAPTER 6 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$K_{rot} = \frac{1}{2} I \omega^2 \mid K_{total} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$	Rotational kinetic energy and total kinetic energy of a rolling body
$v_{cm} = R\omega, a_{cm} = R\alpha \mid v_{bottom} = \sqrt{(2gh / (1 + c))}$	No-slip rolling constraints and incline race speed (where $I_{cm} = cMR^2$)
$W_{rot} = \int \tau(\theta)\, d\theta = \Delta K_{rot} \mid P_{rot} = \tau \omega$	Rotational work-energy theorem and instantaneous rotational power
$\vec{L} = \vec{r} \times \vec{p}\ (\lvert\vec{L}\rvert = rmv \sin\theta = mv r_{\perp}) \mid \vec{L} = I \vec{\omega}$	Angular momentum of a point particle and a rigid body (kg·m²/s)
$\sum \vec{\tau} = d\vec{L}/dt \mid \Delta\vec{L} = \int \vec{\tau}(t)\, dt \mid I_i \omega_i = I_f \omega_f$	Angular impulse-momentum theorem and conservation of angular momentum
$v_{orb} = \sqrt{(GM/r)} \mid v_{esc} = \sqrt{(2GM/R)} \mid E_{tot} = -GMm/(2r) = -K = \frac{1}{2}U_g \mid T^2 = (4\pi^2/GM)\, r^3$	Circular orbital speed, escape speed, orbital energy relations, and Kepler's 3rd Law

§ 6.1 Rotational Kinetic Energy, Rolling Without Slipping & The Incline Race

Even when a flywheel spins in place with fixed center of mass, every particle inside it is moving and carries kinetic energy. Replacing m → I and v → ω in ½mv² yields **rotational kinetic energy: K_rot = ½Iω²** (measured in joules, J). Furthermore, rotational work done by a torque over angular displacement is **W_rot = ∫(θ_i to θ_f) τ(θ) dθ = ΔK_rot**, and instantaneous rotational power is **P_rot = τω**.

When a wheel, cylinder, or sphere **rolls without slipping** along a surface, its motion is the superposition of center-of-mass translation and rotation about the center of mass, locked together by the no-slip constraints: **v_cm = Rω** and **a_cm = Rα**. At any instant, the bottom contact point has **zero velocity relative to the ground (v_bottom = 0)**, the center moves at v_cm, and the top point moves forward at **2v_cm**. Total kinetic energy is the sum of both forms: **K_total = ½Mv_cm² + ½I_cm ω²**.

When a rigid body with moment of inertia **I_cm = cMR²** rolls from rest without slipping down an incline of vertical height h, static friction acts at the bottom contact point to provide torque, but because that contact point has instantaneous velocity v_bottom = 0, **static friction does zero work (W_fs = 0)**! Mechanical energy is conserved: **Mgh = ½Mv² + ½(cMR²)(v/R)² = ½Mv²(1 + c)**. Both M and R cancel out completely, yielding bottom speed **v = √(2gh / (1 + c))**.

- **The Great Rolling Incline Race:** Smaller shape factor c means less energy is tied up in rotation and more goes into forward translation! Thus a frictionless sliding block (c = 0) beats a solid sphere (c = 2/5 = 0.40), which beats a solid disk/cylinder (c = 1/2 = 0.50), which beats a hollow spherical shell (c = 2/3 ≈ 0.67), while a thin hoop (c = 1.0) always finishes last.

Table 6.1 · Rolling Without Slipping Incline Race Results ($I_{cm} = cMR^2$)

Object Rolling Down Height h	Shape Factor c	Linear Acceleration a_{cm}	Final Speed at Bottom v
Frictionless Sliding Block (no rotation)	c = 0	g sin θ	√(2gh) = 1.000 √(2gh)
Uniform Solid Sphere	c = 2/5 = 0.40	(5/7) g sin θ ≈ 0.714 g sin θ	√(10gh / 7) ≈ 0.845 √(2gh)
Uniform Solid Disk / Cylinder	c = 1/2 = 0.50	(2/3) g sin θ ≈ 0.667 g sin θ	√(4gh / 3) ≈ 0.816 √(2gh)
Thin Hollow Spherical Shell	c = 2/3 ≈ 0.67	(3/5) g sin θ = 0.600 g sin θ	√(6gh / 5) ≈ 0.775 √(2gh)
Thin Hoop / Hollow Ring	c = 1 = 1.00	(1/2) g sin θ = 0.500 g sin θ	√(gh) ≈ 0.707 √(2gh)

EXAMPLE 6.1 **Energy Conservation Derivation of the Rolling Race Formula**

Problem: A uniform solid sphere ($I_{cm} = \frac{2}{5}MR^2$) and a thin hoop ($I_{cm} = MR^2$) are released simultaneously from rest at the top of a rough incline of vertical height h and roll without slipping to the bottom. Derive the final speed of each object in terms of g and h.

STEP-BY-STEP SOLUTION:

1. Step 1: Write conservation of mechanical energy for a general rolling body with $I_{cm} = cMR^2$ and $\omega = v/R$: $Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}(cMR^2)(v/R)^2 = \frac{1}{2}Mv^2(1 + c)$.
2. Step 2: Cancel M and solve for v: $v = \sqrt{(2gh / (1 + c))}$. Notice that neither mass M nor radius R affects the outcome.
3. Step 3: Substitute c = 2/5 for the solid sphere: $v_{sphere} = \sqrt{(2gh / (1 + 2/5))} = \sqrt{(10gh / 7)}$. Substitute c = 1 for the thin hoop: $v_{hoop} = \sqrt{(2gh / (1 + 1))} = \sqrt{(gh)}$.

Final Answer: $v_{sphere} = \sqrt{(10gh/7)} > v_{hoop} = \sqrt{(gh)}$ (the solid sphere always wins).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: If a sphere rolls down a FRICTIONLESS incline, there is no static friction to provide torque ($\tau = 0 \Rightarrow \omega = 0$), so the sphere slides without rotating ($K_{rot} = 0$)! Also, if a spinning ball is launched UP a frictionless ramp, its rotational kinetic energy $\frac{1}{2}I\omega^2$ remains constant on the way up and CANNOT convert into gravitational potential energy Mgh.

§ 6.2 Angular Momentum (L = r × p, L = Iω) & Conservation of Angular Momentum

Angular momentum ($\vec{L}$) has two fundamental expressions on the AP Physics C reference sheet: (1) for a **single point particle** moving with linear momentum $\vec{p} = m\vec{v}$, its angular momentum about a reference point is the cross product $\vec{L} = \vec{r} \times \vec{p}$, with magnitude $L = rmv \sin\theta = mv\, r_\perp$, where $r_\perp = r \sin\theta$ is the perpendicular distance of closest approach (impact parameter) from the pivot to the particle's line of motion; and (2) for an **extended rigid body** spinning about a symmetry axis, $\vec{L} = I\vec{\omega}$ (in kg·m²/s). Crucially, a particle moving in a straight line *does* possess constant non-zero angular momentum $L = mv\, r_\perp$ about any pivot not on its line of motion!

Just as net force equals $d\vec{p}/dt$, net external torque equals the time derivative of angular momentum: $\sum\vec{\tau} = d\vec{L}/dt$. Integrating with respect to time gives the **Angular Impulse-Momentum Theorem**: $\Delta\vec{L} = \int_{t_1}^{t_2} \vec{\tau}(t)\, dt = \vec{L}_f - \vec{L}_i$, which equals the signed area under a Torque vs. Time graph τ(t).

Whenever the net external torque on a system about a fixed axis is zero ($\sum\vec{\tau}_{ext} = 0$), $d\vec{L}/dt = 0$ and **total angular momentum is strictly conserved: $\vec{L}_{initial} = \vec{L}_{final}$ (or $I_i \omega_i = I_f \omega_f$)**. When a spinning figure skater pulls her arms inward, reducing her rotational inertia I by a factor of 3 with zero external torque, her angular speed ω triples to keep Iω constant. Writing rotational kinetic energy as $K_{rot} = \frac{1}{2}I\omega^2 = L^2/(2I)$ shows that reducing I by a factor of 3 while L is constant *triples* her rotational kinetic energy (paid for by internal muscular work pulling her arms inward).

- **Point-Mass Straight-Line L:** As a ball flies in a straight line past a pivot, r and θ change continuously, but the perpendicular lever arm $r_\perp = r \sin\theta$ remains constant, keeping $L = mv\, r_\perp$ constant.

EXAMPLE 6.2 Coaxial Two-Disk Rotational Collision

Problem: Two separated horizontal disks spin frictionlessly on the same central vertical axle. Taking counterclockwise (CCW) as positive (+), the top disk has rotational inertia $I_1 = 12\ \text{kg·m}^2$ and angular velocity $\omega_1 i = +6.0$ rad/s (CCW), while the bottom disk has $I_2 = 10\ \text{kg·m}^2$ and spins clockwise at $\omega_2 i = -2.0$ rad/s (CW). The top disk drops onto the bottom disk and friction locks them together. Find their common final angular velocity ω_f.

STEP-BY-STEP SOLUTION:

1. Step 1 (Initial Angular Momentum): $L_i = I_1\omega_1 i + I_2\omega_2 i = (12)(+6.0) + (10)(-2.0) = +72 - 20 = +52\ \text{kg·m}^2/\text{s}$.
2. Step 2 (Combined Rotational Inertia): After locking together on the same axle, $I_{total} = I_1 + I_2 = 12 + 10 = 22\ \text{kg·m}^2$.
3. Step 3 (Conservation of Angular Momentum): Since frictional torque between the disks is internal to the two-disk system ($\sum\tau_{ext} = 0$), $L_f = L_i \Rightarrow 22\ \omega_f = +52 \Rightarrow \omega_f = 52 / 22 \approx +2.36$ rad/s (CCW).

Final Answer: $\omega_f = +26/11$ rad/s $\approx +2.36$ rad/s (counterclockwise).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Always include directional signs (+ for CCW, - for CW) when summing angular momenta! Had you added 72 + 20 = 92 instead of 72 - 20 = 52, you would get the wrong answer.

§ 6.3 Linear-to-Rotational Collisions: Ball Striking a Pivoted Rod

One of the classic multi-part AP Physics C free-response problems involves a point mass m moving horizontally at speed v_0 that strikes and sticks to the bottom tip of a vertical uniform rod of mass M, length L, and rotational inertia $I_{rod} = \frac{1}{3}ML^2$ pivoted at its top end.

During the impact: (1) **Linear momentum is NOT conserved** because the top hinge exerts an external horizontal impulse on the rod to prevent the pivot from moving; (2) **Mechanical energy is NOT conserved** because the collision is inelastic (sticking); (3) **Angular momentum about the top pivot hinge IS conserved** because the external hinge force acts at r = 0 and produces zero torque! Equating initial point-mass angular momentum $L_i = L\ m\ v_0$ (since r = L and θ = 90°) to final rigid-body angular momentum $L_f = I_{total}\ \omega_f = (I_{rod} + mL^2)\ \omega_f$ determines ω_f immediately after impact.

During the subsequent upward swing of the combined rod-plus-ball system, mechanical energy *is* conserved: **$\frac{1}{2}\ I_{total}\ \omega_f^2 = (M + m)g\ \Delta y_{cm}$**, where $\Delta y_{cm} = r_{cm}(1 - \cos\theta)$ is the vertical rise of the **combined system's center of mass**.

- **Combined Center of Mass Rule:** When computing gravitational potential energy gain $\Delta U_g = M_{total}\ g\ \Delta y_{cm}$ for a composite rigid body (rod + stuck ball), you must track the vertical rise of the combined center of mass (or sum $M_{rod}\ g\ \Delta y_{rod,cm} + m_{ball}\ g\ \Delta y_{ball}$ separately).

EXAMPLE 6.3 **Full FRQ Synthesis: Ball Sticking to a Pivoted Rod & Subsequent Swing**

Problem: A 1.0 kg ball moving horizontally at v_0 = 5.0 m/s strikes the bottom tip (r = 1.0 m) of a stationary 4.0 kg vertical uniform rod of length L = 1.0 m and top-hinge rotational inertia I_rod = 4/3 kg·m^2, and sticks to it (use g = 10 m/s^2). Find (a) the angular speed ω_f immediately after impact, and (b) how high the ball rises at the peak of the swing.

STEP-BY-STEP SOLUTION:

1. Step 1 (Part a — Angular Momentum Conservation about Top Hinge): Initial angular momentum of the ball about the top hinge is L_i = r m v_0 sin 90° = (1.0 m)(1.0 kg)(5.0 m/s) = 5.0 kg·m^2/s.
2. Step 2 (Part a — Total Inertia & ω_f): After sticking, I_total = I_rod + m_ball r^2 = 4/3 + (1.0)(1.0)^2 = 7/3 kg·m^2. Setting L_i = I_total ω_f gives 5.0 = (7/3) ω_f ⇒ ω_f = 15/7 ≈ 2.14 rad/s.
3. Step 3 (Part b — Kinetic Energy After Impact): K_rot = ½ I_total ω_f^2 = ½(7/3)(15/7)^2 = 75/14 ≈ 5.36 J (or 5.34 J using 2.14 rad/s).
4. Step 4 (Part b — Combined Center of Mass & Swing Height): Measuring upward from the bottom of the rod, y_cm,i = [m_ball(0) + M_rod(0.50)] / (1.0 + 4.0) = 2.0 / 5.0 = 0.40 m above the bottom (so distance from the top pivot to the combined CM is d_cm = 0.60 m = 3/5 m). Setting K_rot = M_total g Δy_cm gives 5.34 = (5.0)(10) Δy_cm ⇒ Δy_cm = 0.107 m. Since the ball is at radius 1.0 m while the CM is at radius 0.60 m from the pivot, by similar arcs the ball rises Δy_ball = (1.0 / 0.60) Δy_cm = (5/3)(0.107 m) ≈ 0.18 m.

Final Answer: (a) ω_f = 15/7 rad/s ≈ 2.14 rad/s; (b) Δy_cm = 0.107 m and ball rise Δy_ball ≈ 0.18 m.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never use linear momentum conservation (mv_0 = (m+M)v) when a ball hits a rod that is pinned at a hinge! The hinge exerts an external horizontal force, so only angular momentum about the hinge is conserved.

§ 6.4 Orbital Mechanics, Escape Velocity & Kepler's Three Laws

For a satellite of mass m in a **circular orbit** of radius r around a central host body of mass M, universal gravitation provides the required inward centripetal force: **GMm/r^2 = mv_orb^2/r**. Canceling satellite mass m and one power of r yields the circular orbital speed: **v_orb = √(GM/r)**. Higher-altitude orbits (larger r) have slower orbital speeds.

Multiplying mv_orb^2 = GMm/r by ½ reveals the orbital kinetic energy: **K = +GMm/(2r)**. Adding universal gravitational potential energy **U_g = -GMm/r** gives the total mechanical energy of a circular orbit: **E_total = K + U_g = GMm/(2r) - GMm/r = -GMm/(2r)**. Memorize this golden virial relationship for any circular orbit: **E_total = -K = ½U_g** (and U_g = -2K). The negative sign of E_total signifies a gravitationally bound orbit.

Escape velocity (v_esc) is the minimum initial speed needed to launch an object from radius R so that it reaches r = ∞ with zero leftover speed (K_f = 0 and U_gf = 0, so E_total = 0). Setting ½m v_esc^2 - GMm/R = 0 yields **v_esc = √(2GM/R) = √2 v_orb**. Finally, **Kepler's Three Laws** govern all planetary and satellite orbits: (1) **Law of Orbits**: planets move in ellipses with the central star at one focus; (2) **Law of Areas**: a line from the star to the planet sweeps out equal areas in equal times—a direct consequence of **conservation of angular momentum** (since radial gravity acts along r⃗, τ⃗ = r⃗ × F⃗_g = 0, so L = mvr sin θ is constant; at perihelion and aphelion where θ = 90°, **v_p r_p = v_a r_a**); and (3) **Law of Periods**: substituting v_orb = 2πr/T into v_orb^2 = GM/r yields **T^2 = (4π^2/GM) r^3**.

- **Bound vs. Unbound Trajectories:** For E_total < 0, the orbit is bound (circle or ellipse); for E_total = 0 (v = v_esc), the trajectory is a parabolic escape path; for E_total > 0 (v > v_esc), the object escapes along a hyperbola with leftover speed at infinity.

EXAMPLE 6.4 Escape Energy Deficit & Perihelion-to-Aphelion Speed Ratio

Problem: (a) A rocket resting on a planet's surface sits in a gravitational potential energy well of $U_g = -40$ MJ. How much initial kinetic energy must be supplied to barely escape to infinity, and how does v_{esc} compare to surface circular orbital speed? (b) A comet in an elliptical orbit around the Sun has perihelion distance r_p and aphelion distance $r_a = 4r_p$. Find the ratio of its perihelion speed v_p to its aphelion speed v_a.

STEP-BY-STEP SOLUTION:

1. Step 1 (Part a): At $r = \infty$ with zero final speed, $E_{total} = K_f + U_{gf} = 0$. Thus $K_i + U_{gi} = 0 \Rightarrow K_i = -U_{gi} = +40$ MJ. Comparing $v_{esc} = \sqrt{(2GM/R)}$ to $v_{orb} = \sqrt{(GM/R)}$ shows $v_{esc} = \sqrt{2}\, v_{orb}$.
2. Step 2 (Part b): Because gravitational force is radial (parallel to r), gravitational torque about the Sun is zero ($\tau = 0$), so angular momentum L is conserved throughout the elliptical orbit.
3. Step 3 (Part b): At perihelion and aphelion, velocity v is perpendicular to r ($\sin 90° = 1$), so $m\, v_p\, r_p = m\, v_a\, r_a \Rightarrow v_p / v_a = r_a / r_p = 4r_p / r_p = 4$.

Final Answer: (a) $K_i = +40$ MJ and $v_{esc} = \sqrt{2}\, v_{orb}$; (b) $v_p / v_a = 4$.

⚠ AP EXAM TRAP & FRQ SCORING GUIDELINE

AP Exam Trap: When moving a satellite from a lower circular orbit r_1 to a higher circular orbit r_2, orbital speed $v_{orb} = \sqrt{(GM/r)}$ DECREASES (so K decreases), but total mechanical energy $E_{total} = -GMm/(2r)$ becomes less negative (so E_{total} INCREASES, requiring positive thruster work)!

✅ CHAPTER 6 MASTERY CHECKLIST (AP EXAM READINESS)

☑ Apply rolling without slipping ($v_{cm} = R\omega$, $K_{total} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$) and rank incline race speeds via $v = \sqrt{(2gh/(1+c))}$.

☑ Calculate point-particle angular momentum $L = mv\, r_\perp$ and rigid-body angular momentum $L = I\omega$.

☑ Conserve angular momentum about a fixed hinge during rod-ball collisions, then conserve mechanical energy during the swing.

☑ Derive $v_{orb} = \sqrt{(GM/r)}$, $E_{tot} = -GMm/(2r)$, $v_{esc} = \sqrt{(2GM/R)}$, Kepler's 2nd Law ($v_p\, r_p = v_a\, r_a$), and Kepler's 3rd Law ($T^2 \propto r^3$).

CHAPTER 7 · CORE CURRICULUM

AP Exam Weighting: 10–15%

Unit 7: Oscillations & Simple Harmonic Motion

Simple Harmonic Motion (SHM) occurs whenever a system displaced from stable equilibrium experiences a linear restoring force or torque proportional to displacement. This final unit covers the canonical SHM differential equation $d^2x/dt^2 + \omega^2 x = 0$, sinusoidal solutions and calculus phase derivatives, horizontal and vertical mass-spring systems, series and parallel spring combinations, parabolic energy graphs, and simple, physical, and torsional pendulums.

📐 CHAPTER 7 ESSENTIAL EQUATIONS, LAWS & CONSTANTS

$d^2x/dt^2 + \omega^2 x = 0$ (where $\omega^2 = k/m$ for a mass-spring)	Canonical differential equation of Simple Harmonic Motion (SHM)
$x(t) = A\cos(\omega t + \varphi)$ | $v(t) = -\omega A \sin(\omega t + \varphi)$ | $a(t) = -\omega^2 A \cos(\omega t + \varphi) = -\omega^2 x$	Sinusoidal position, velocity, and acceleration in SHM ($v_{max} = \omega A$, $a_{max} = \omega^2 A$)
$T = 1/f = 2\pi/\omega$ | $T_s = 2\pi\sqrt{(m/k)}$	Period, frequency, angular frequency, and mass-spring period (independent of A)
$E_{total} = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$	Conservation of mechanical energy in an undamped harmonic oscillator
$T_p = 2\pi\sqrt{(L/g)}$ | $T_{phys} = 2\pi\sqrt{(I/(mgd))}$ | $T_{tors} = 2\pi\sqrt{(I/\kappa)}$	Periods of simple, physical, and torsional pendulums (small-angle approximation)

§ 7.1 Linear Restoring Force, SHM Differential Equation & Period

Periodic motion is any motion that repeats its path in equal time intervals. Before analyzing oscillators, note three standard AP Physics C free-response wording conventions: (1) '*smooth surface*' means negligible friction, whereas '*rough surface*' means friction is present; (2) never assume air resistance unless explicitly stated; and (3) strings and cords are assumed massless unless stated otherwise. Periodic motion is specifically **Simple Harmonic Motion (SHM)** whenever the net restoring force is strictly **linear in displacement from equilibrium and directed back toward equilibrium: $F_s = -kx$**.

To derive the **differential equation of SHM** for a block of mass m attached to a spring of constant k on a smooth horizontal surface, start with Newton's Second Law: $F_{net} = ma = -kx$. Replacing acceleration a with the second time derivative d^2x/dt^2 and dividing by m yields: **$d^2x/dt^2 = -(k/m)x$**, or **$d^2x/dt^2 + \omega^2 x = 0$**. Whenever any physical system obeys **$d^2u/dt^2 = -(\text{constant})\cdot u$**, the motion is SHM and that positive constant is always **ω^2**.

The **period (T)** is the time in seconds for one complete cycle (returning to the exact same position *and* velocity direction). Frequency is **$f = 1/T$** (in hertz, Hz = s^{-1}), and angular frequency is **$\omega = 2\pi/T = 2\pi f$** (in rad/s). Equating $T = 2\pi/\omega$ to the reference-sheet spring period **$T_s = 2\pi\sqrt{(m/k)}$** confirms that **$\omega = \sqrt{(k/m)}$** (or $\omega^2 = k/m$). Remarkably, **amplitude A does not appear anywhere in $T_s = 2\pi\sqrt{(m/k)}$**: pulling a spring twice as far increases the block's peak speed proportionally so that one full cycle takes the exact same time!

- **The Equilibrium Half-Period Trap:** Starting at equilibrium x = 0 moving left, reaching -A, and returning to x = 0 moving right is only HALF a period (T/2)! The block must go all the way to +A and return to x = 0 moving left to complete one full period T.
- **Quarter-Period Amplitude Travel:** Moving from x = +A to x = 0 takes T/4; moving from x = +A to x = -A takes T/2; moving one full cycle takes T (total distance 4A).

EXAMPLE 7.1 **Identifying Angular Frequency & Period from a Second-Order ODE**

Problem: A particle moving along the x-axis obeys the differential equation $3(d^2x/dt^2) + 48x = 0$, where x is in meters and t is in seconds. Determine (a) the angular frequency ω, (b) the period T, and (c) the frequency f of the oscillation.

STEP-BY-STEP SOLUTION:

1. Step 1: Divide the differential equation by 3 to put it into canonical form $d^2x/dt^2 + \omega^2 x = 0$: $d^2x/dt^2 + 16x = 0$.
2. Step 2: Identify $\omega^2 = 16\ s^{-2} \Rightarrow \omega = 4.0$ rad/s.
3. Step 3: Compute period $T = 2\pi / \omega = 2\pi / 4.0 = \pi/2$ s ≈ 1.57 s, and frequency $f = 1/T = 2/\pi$ Hz ≈ 0.637 Hz.

Final Answer: ω = 4.0 rad/s, T = π/2 s (≈ 1.57 s), and f = 2/π Hz (≈ 0.637 Hz).

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never write the SHM differential equation as $d^2x/dt^2 = +\omega^2 x$! Without the minus sign (restoring direction), the solution is exponential runaway (e^(ωt)), not sinusoidal oscillation.

§ 7.2 Sinusoidal Solutions, Calculus Derivatives & Phase Relationships

Which mathematical functions have a second derivative equal to $-\omega^2$ times themselves? Sine and cosine! On the AP Physics C reference sheet, the general solution to $d^2x/dt^2 = -\omega^2 x$ is written as **$x(t) = x_{max} \cos(\omega t + \varphi) = A \cos(\omega t + \varphi)$**, where **$A = x_{max}$** is the amplitude (maximum displacement from equilibrium) and **φ** is the phase constant determined by initial conditions at t = 0 (use cos(ωt) when released from rest at x = +A, or ±sin(ωt) when launched from equilibrium x = 0).

Applying the chain rule of calculus once and twice to $x(t) = A \cos(\omega t + \varphi)$ yields instantaneous velocity and acceleration: **$v(t) = dx/dt = -\omega A \sin(\omega t + \varphi)$** (with maximum speed **$v_{max} = \omega A$**) and **$a(t) = dv/dt = -\omega^2 A \cos(\omega t + \varphi) = -\omega^2 x(t)$** (with maximum acceleration magnitude **$a_{max} = \omega^2 A = kA/m$**).

Comparing x(t), v(t), and a(t) reveals the essential phase relationships tested on every AP exam: at the **turning points ($x = \pm A$)**, the oscillator momentarily stops to reverse direction (**$v = 0$**) while the spring is maximally stretched or compressed, so restoring force and acceleration magnitude are at their **absolute maximum ($|a| = a_{max} = \omega^2 A$)**. Conversely, as the oscillator flies through **equilibrium ($x = 0$)**, spring force and acceleration vanish (**$F = 0$, $a = 0$**) while speed reaches its **absolute maximum ($|v| = v_{max} = \omega A$)**.

- **90° and 180° Phase Shifts:** Velocity v(t) leads position x(t) by 90° (π/2 rad, quarter cycle), while acceleration $a(t) = -\omega^2 x(t)$ is 180° (π rad, half cycle) out of phase with position.

Table 7.1 · Instantaneous Values of SHM Variables Across Key Positions

Oscillator Position	Displacement x	Speed |v|	Acceleration a	Kinetic Energy K	Potential Energy U_s
Left Turning Point	-A	0 (minimum)	$+\omega^2 A$ (max rightward)	0	$\frac{1}{2}kA^2$ (maximum)
Equilibrium Center	0	$v_{max} = \omega A$ (max)	0 (minimum)	$\frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2$ (max)	0 (minimum)
Right Turning Point	+A	0 (minimum)	$-\omega^2 A$ (max leftward)	0	$\frac{1}{2}kA^2$ (maximum)
Equal-Energy Point	$\pm A/\sqrt{2}$ ($\approx \pm 0.707A$)	$v_{max}/\sqrt{2}$	$\mp\omega^2 A/\sqrt{2}$	$\frac{1}{4}kA^2$ (½ E_tot)	$\frac{1}{4}kA^2$ (½ E_tot)
Half-Amplitude Point	$\pm A/2$	$(\sqrt{3}/2)\, v_{max}$ ($\approx 0.866\, v_{max}$)	$\mp\omega^2 A/2$	¾ E_tot (75%)	¼ E_tot (25%)

EXAMPLE 7.2 **Extracting Kinematics & Mass from an SHM Position Equation**

> **Problem:** A block attached to a spring with stiffness k = 200 N/m oscillates on a frictionless horizontal table according to $x(t) = 0.30\cos(10t)$ m. Find (a) the amplitude A and angular frequency ω, (b) the mass m of the block, and (c) the block's maximum speed v_{max} and maximum acceleration a_{max}.

STEP-BY-STEP SOLUTION:

1. Step 1: Match $x(t) = 0.30\cos(10t)$ to $A\cos(\omega t + \varphi)$: A = 0.30 m and ω = 10 rad/s (with φ = 0).
2. Step 2: Use $\omega^2 = k/m$ to find mass m: $10^2 = 200/m \Rightarrow 100 = 200/m \Rightarrow m = 2.0$ kg.
3. Step 3: Compute maximum speed $v_{max} = \omega A = (10\text{ rad/s})(0.30\text{ m}) = 3.0$ m/s, and maximum acceleration $a_{max} = \omega^2 A = (10\text{ rad/s})^2(0.30\text{ m}) = 30\text{ m/s}^2$.

Final Answer: **(a) A = 0.30 m, ω = 10 rad/s; (b) m = 2.0 kg; (c) v_{max} = 3.0 m/s, $a_{max} = 30\text{ m/s}^2$.**

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: When differentiating $x(t) = A\cos(\omega t)$, students frequently forget the chain rule factor of ω! Differentiating $\cos(\omega t)$ pulls out one factor of ω for velocity ($v_{max} = \omega A$) and a second factor of ω for acceleration ($a_{max} = \omega^2 A$).

§ 7.3 Energy in SHM, Parabolic Energy Graphs, Vertical & Multi-Spring Systems

On a frictionless surface, total mechanical energy in SHM is strictly conserved: $\mathbf{E_{total} = U_s + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2}$. Evaluating E_{total} at the turning points ($x = \pm A$, $v = 0$) gives $\mathbf{E_{total} = \frac{1}{2}kA^2}$; evaluating at equilibrium ($x = 0$, $v = v_{max}$) gives $\mathbf{E_{total} = \frac{1}{2}mv_{max}^2}$. At any intermediate position x, $\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$ allows immediate calculation of speed: $\mathbf{v(x) = \pm\omega\sqrt{A^2 - x^2}}$.

Plotted against position x from -A to +A, spring potential energy $\mathbf{U_s(x) = \frac{1}{2}kx^2}$ is an upward-opening parabola, kinetic energy $\mathbf{K(x) = \frac{1}{2}k(A^2 - x^2)}$ is a downward-opening parabola, and total energy $E_{total} = \frac{1}{2}kA^2$ is a flat horizontal line. The two parabolas intersect where $\mathbf{U_s = K = \frac{1}{2}E_{total}}$, which occurs at $\mathbf{x = \pm A/\sqrt{2} \approx \pm 0.707A}$ (NOT at $x = \pm A/2$, where $U_s = \frac{1}{4}E_{total}$ and $K = \frac{3}{4}E_{total}$!). Plotted against time t, both K(t) and $U_s(t)$ oscillate between 0 and $\frac{1}{2}kA^2$ at **twice the frequency (2f)** and half the period (T/2) of position x(t).

When a mass m is hung from a **vertical spring** of constant k, gravity stretches the spring to a new static equilibrium position $\mathbf{y_{eq} = mg/k}$ (where upward ky_{eq} balances downward mg). Measuring displacement y' from this shifted hanging equilibrium cancels the constant weight mg against ky_{eq}, leaving net restoring force $\mathbf{F_{net} = -ky'}$. Thus a vertical mass-spring system oscillates about its shifted equilibrium with the **exact same period $\mathbf{T_s = 2\pi\sqrt{(m/k)}}$**! For a

block connected to two springs, **parallel springs** (side-by-side or on opposite sides of a block between two walls) stretch by the same x so stiffnesses add (**$k_{eff} = k_1 + k_2$**, shorter period), whereas **series springs** (end-to-end) share the same tension so reciprocals add (**$1/k_{eff} = 1/k_1 + 1/k_2$**, longer period).

- **Amplitude Doubling Rule:** Because $E_{total} = \frac{1}{2}kA^2$, doubling the amplitude A of an oscillator quadruples its total mechanical energy ($4E_{total}$), doubles its maximum speed ($2v_{max}$), doubles its maximum acceleration ($2a_{max}$), and leaves the period T completely unchanged!

EXAMPLE 7.3 **Finding Where Kinetic and Potential Energies Are Equal in SHM**

> **Problem:** A mass-spring oscillator vibrates with amplitude A and total mechanical energy $E_{total} = \frac{1}{2}kA^2$. (a) At what displacement x (in terms of A) does kinetic energy equal elastic potential energy ($K = U_s$)? (b) What fraction of total energy is kinetic when $x = \frac{1}{2}A$?

STEP-BY-STEP SOLUTION:

1. Step 1 (Part a): Set $U_s = K \Rightarrow$ Since $U_s + K = E_{total}$, we have $U_s = \frac{1}{2}E_{total} \Rightarrow \frac{1}{2}kx^2 = \frac{1}{2}(\frac{1}{2}kA^2)$.
2. Step 2 (Part a): Cancel $\frac{1}{2}k$ and take the square root: $x^2 = A^2/2 \Rightarrow x = \pm A/\sqrt{2} \approx \pm 0.707A$.
3. Step 3 (Part b): At $x = \frac{1}{2}A$, potential energy is $U_s = \frac{1}{2}k(A/2)^2 = \frac{1}{4}(\frac{1}{2}kA^2) = \frac{1}{4}E_{total}$. Therefore $K = E_{total} - U_s = \frac{3}{4}E_{total}$ (75% kinetic, 25% potential).

Final Answer: (a) $x = \pm A/\sqrt{2}$; (b) $K = \frac{3}{4}E_{total}$ and $U_s = \frac{1}{4}E_{total}$ at $x = A/2$.

⚠ **AP EXAM TRAP & FRQ SCORING GUIDELINE**

AP Exam Trap: Never say $K = U_s$ at $x = A/2$! Because potential energy $U_s = \frac{1}{2}kx^2$ depends on x SQUARED, at $x = A/2$ potential energy is only $(1/2)^2 = 1/4$ of total energy; $K = U_s$ occurs at $x = \pm A/\sqrt{2}$.

§ 7.4 Simple, Physical & Torsional Pendulums

A **simple pendulum** consists of a point mass m (bob) suspended from a massless string of length L. When displaced by angle θ from the vertical, string tension T balances radial gravity mg cos θ, while the tangential component of gravity pulls the bob back toward equilibrium: **$F_{restoring} = -mg \sin \theta$**. Because sin θ is not linear in θ, a pendulum is not strictly SHM at large angles; however, for **small angles ($\theta \lesssim 15°$)** when θ is measured in radians, the **small-angle approximation $\sin \theta \approx \theta$** makes the restoring force linear! Dividing by m and using arc length $s = L\theta$ (so $a_t = d^2s/dt^2 = L\, d^2\theta/dt^2$) yields: **$d^2\theta/dt^2 + (g/L)\theta = 0$**. Matching $\omega^2 = g/L$ gives the simple pendulum period: **$T_p = 2\pi\sqrt{(L/g)}$**, completely independent of bob mass m and small amplitude θ_{max}.

When an extended rigid body of mass m and rotational inertia I (about the pivot) is pivoted at distance **d from its center of mass**, it forms a **physical pendulum**—a signature topic exclusive to AP Physics C. Gravity acts at the center of mass to produce restoring torque **$\tau = -mgd \sin \theta \approx -mgd\,\theta$**. Setting $\tau = I\alpha = I(d^2\theta/dt^2)$ yields **$d^2\theta/dt^2 + (mgd/I)\theta = 0$**, with $\omega^2 = mgd/I$ and period **$T_{phys} = 2\pi\sqrt{(I / (mgd))}$**.

Finally, if a disk of rotational inertia I hangs from a wire that exerts a linear restoring torque **$\tau = -\kappa\theta$** when twisted (where κ is the torsion constant in N·m/rad), setting $I(d^2\theta/dt^2) = -\kappa\theta$ yields a **torsional pendulum** with $\omega^2 = \kappa/I$ and period **$T_{tors} = 2\pi\sqrt{(I/\kappa)}$**. Notice the universal pattern across all four harmonic oscillators: every period equals **T = 2π√(Inertia Factor / Restoring Stiffness Factor)**.

- **Universal Oscillator Period Pattern:** Every SHM period has the structure $T = 2\pi / \omega = 2\pi\sqrt{(\text{Inertia} / \text{Stiffness})}$: Mass-Spring $\sqrt{(m/k)}$, Simple Pendulum $\sqrt{(L/g)}$, Physical Pendulum $\sqrt{(I/mgd)}$, and Torsional Pendulum $\sqrt{(I/\kappa)}$.

Table 7.2 · Master Comparison of All Four AP Physics C Harmonic Oscillators

Oscillator Type	Restoring Law	Canonical Differential Equation	Angular Frequency ω	Period T = 2π/ω
Mass-Spring (Horizontal/Vertical)	$F_x = -k x$	$d^2x/dt^2 + (k/m) x = 0$	$\omega = \sqrt{(k / m)}$	$T_s = 2\pi \sqrt{(m / k)}$
Simple Pendulum (small θ)	$F_t \approx -mg \theta$	$d^2\theta/dt^2 + (g/L) \theta = 0$	$\omega = \sqrt{(g / L)}$	$T_p = 2\pi \sqrt{(L / g)}$
Physical Pendulum (small θ)	$\tau \approx -mgd \theta$	$d^2\theta/dt^2 + (mgd / I) \theta = 0$	$\omega = \sqrt{(mgd / I)}$	$T_{phys} = 2\pi \sqrt{(I / (mgd))}$
Torsional Pendulum	$\tau = -\kappa \theta$	$d^2\theta/dt^2 + (\kappa / I) \theta = 0$	$\omega = \sqrt{(\kappa / I)}$	$T_{tors} = 2\pi \sqrt{(I / \kappa)}$

EXAMPLE 7.4 **Period of a Uniform Pivoted Rod Swinging as a Physical Pendulum**

> **Problem:** A uniform thin metal rod of mass M and length L is pivoted frictionlessly at one end (where $I_{end} = \frac{1}{3}ML^2$) and displaced by a small angle θ from the vertical. Derive the period T_{rod} of its small oscillations and compare it to a simple pendulum of length L.

STEP-BY-STEP SOLUTION:

1. Step 1: Identify the physical pendulum parameters for the uniform rod pivoted at one end: rotational inertia $I = \frac{1}{3}ML^2$, and distance from pivot to the rod's center of mass is $d = L/2$.
2. Step 2: Substitute $I = \frac{1}{3}ML^2$ and $d = L/2$ into the physical pendulum period formula $T_{phys} = 2\pi \sqrt{(I / (Mgd))}$: $T_{rod} = 2\pi \sqrt{[(\frac{1}{3}ML^2) / (Mg(L/2))]}$.
3. Step 3: Cancel M and one power of L: $T_{rod} = 2\pi \sqrt{(2L / (3g))} = \sqrt{(2/3)}\, T_{simple} \approx 0.816\, T_{simple}$ (the rod swings with an effective length $L_{eff} = \frac{2}{3}L$).

Final Answer: $T_{rod} = 2\pi \sqrt{(2L / (3g))}$ (shorter period than a simple pendulum of length L).

⚠ AP EXAM TRAP & FRQ SCORING GUIDELINE

AP Exam Trap: In the physical pendulum formula $T = 2\pi\sqrt{(I/(mgd))}$, d is the distance from the pivot to the CENTER OF MASS ($d = L/2$ for a uniform rod pivoted at its end), NOT the full length L of the rod!

✅ CHAPTER 7 MASTERY CHECKLIST (AP EXAM READINESS)

☑ Put any linear restoring force/torque equation into canonical SHM form $d^2u/dt^2 + \omega^2 u = 0$ to read off ω and $T = 2\pi/\omega$.

☑ Differentiate $x(t) = A \cos(\omega t + \varphi)$ using the chain rule to find v(t) and a(t), and identify where $|v|$ and $|a|$ are maximum/zero.

☑ Apply $E_{total} = \frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$ and distinguish $x = \pm A/\sqrt{2}$ (where $K = U_s$) from $x = \pm A/2$ (where $K = 3U_s$).

☑ Derive and apply the period formulas for simple pendulums ($2\pi\sqrt{(L/g)}$), physical pendulums ($2\pi\sqrt{(I/mgd)}$), and torsional pendulums ($2\pi\sqrt{(I/\kappa)}$).